MONTHLY NOTICES

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ROYAL ASTRONOMICAL SOCIETY.

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Dr. J. H. JEANS, PRESIDENT, in the Chair.

Alampallam Aiyapattai Narayana Aiyar, Kodaikanal Observatory, South India ;

Jaroslav Císar, D.Sc. (Prague), Secretary, Czechoslovak Legation, London, and 42 Ambrose Avenue, Golders Green, London, N.W. 11 ;

Angarai Seshiah Ganesan, 44 Lillie Road, West Brompton, London, S.W. 6 ;

Robert Edward Taylor, 18 Avon Road, Keynsham, near Bristol ; and

J. F. B. Wardale, B.A., 35 St. George's Square, Westminster, S.W. 1,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Eustace Vivian Bacon, M.B.E., M.A., 51 Drayton Gardens, London, S.W. 10 (proposed by D. Brunt) ;

David Dietz, 1146 Cleveland Heights Boulevard, Cleveland Heights, Ohio, U.S.A. (proposed by A. Swasey) ;

William Hodge, F.R.C.O., 2 Newton Vale, Chapeltown Road, Leeds (proposed by E. Hawks) ; and

W. K. Green, Ph.D., Director of the Observatory of Amherst College, Amherst, Mass., U.S.A. (proposed by F. Slocum).

Julius Bauschinger, Director of the Observatory, Leipzig, Germany ; and

William S. Eichelberger, Naval Observatory, Washington, U.S.A.,

were balloted for and duly elected Associates of the Society.

Sixty-seven presents were announced as having been received since the last meeting, including, amongst others :—

J. Bauschinger, Über die Laplacesche Methode der Bahnbestimmung im Vergleich zur Gauss'schen.

Moscow, Observatoire Astronomique : Annales, Series ii., vol. 7.

ADDRESS

Delivered by the President, Dr. J. H. Jeans, on presenting the Gold Medal of the Society to Sir Frank Watson Dyson, Astronomer Royal, for his contributions to Astronomy in general, and, in particular, for his work on the Proper Motions of the Stars.

It has not been the common practice of the Society to award its Gold Medal to Astronomers Royal, indeed, only one holder of the Greenwich Office has been so honoured before, namely, Airy, who received the medal first in 1833 and again in 1846, not to mention receiving one of the testimonials in 1848 which fell in showers, alike on the just and the unjust, on the discoverers of Neptune, on those who might have discovered Neptune if they had been more alert, and on many at least who could never by any chance have discovered Neptune. The reason why Astronomers Royal seldom achieve medals is not difficult to discover ; it was expressed with the utmost clearness from this Chair on the last occasion on which the medal was presented to a Greenwich Astronomer Royal. In presenting the medal to Airy in 1846, Captain Smyth said : " Our medal was primarily instituted as a mark of approbation on individual exertion, on labours of love ; and not to note our sense of the official merits of public men, or of the rectitude and ability with which they may acquit themselves in their respective offices."

An Astronomer Royal must of necessity be a busy man ; it is naturally somewhat rare to find one who, after the accomplishment of his heavy routine duties, has either the energy or the inclination for astronomical labours of love. It is because the present occupant of the office is such a one that it is my pleasant duty to present the medal to him to-day.

Our Medallist's first recorded contact with astronomy occurred in 1888 when he was awarded half of a Sheepshanks Astronomical Exhibition on the certainly not onerous condition that he should become a member of his own college. His first published paper, " On the Motion of a Satellite round a Spheroidal Planet " (*Quart. Journ. Math.*, vol. xxv., 1891), suggests considerable astronomical knowledge and interest ; his next paper, the well-known investigation " On the Potential of an Anchor Ring " (*Phil. Trans.*, vol. clxxxiv., 1893), apart from having obvious astronomical applications, showed that he was a mathematician of no mean order.

The year following the publication of this last paper saw him appointed Chief Assistant at Greenwich, in immediate succession to the previous Sheepshanks Exhibitioner, Professor H. H. Turner. The work immediately awaiting him was that of assisting in the development of methods for the measurement and reduction of the plates of that troublesome infant, as it then was, the *Astrographic Catalogue*; this brought him up at once against the want of reliable proper motions and so started him on what was to become the central work of his astronomical career.

The only way of determining a proper motion is by comparing the position of a star in the sky now with its position at some previous time. Most stars move so slowly across the sky that a reliable determination of proper motion requires a good determination of the star's present position, about which there is no difficulty, and also a good determination of the star's position at some time in the quite remote past, about which there is generally extreme difficulty. When Dyson went to Greenwich in 1894, the only reliable proper motions were those which had been deduced from Bradley's observations of 1750–1762. These observations, like all made by Bradley, were extraordinarily accurate for their time, but having been made with telescopes of only about $1\frac{1}{2}$ inches aperture, were limited to stars brighter than the seventh magnitude. With a view to getting reliable proper motions of fainter stars, the new Chief Assistant first set his hand, in collaboration with W. G. Thackeray, to a re-reduction of the observations made by Groombridge in the years 1806 to 1816. Stephen Groombridge after a varied career, first as a linen-draper and afterwards as a West Indian merchant, settled down in Blackheath when in his fifty-first year to observe stars. He was gifted with enthusiasm, skill, and a good telescope, the combination of which resulted in some 27,000 observations of 4239 stars lying within 52° of the North Pole. On the average these stars were of seventh magnitude, although some were as faint as the ninth. Thus they supplemented the Bradley observations in a most valuable way. After re-reducing Groombridge's observations, Dyson and Thackeray determined the proper motions of the whole of the 4239 stars, generally by comparison with the latest Greenwich observations, although comparisons were also made with the *Radcliffe Catalogue* of 1845, and the various catalogues of the Astronomische Gesellschaft. The result of their work was published in 1905 in a volume entitled *New Reduction of Groombridge's Catalogue of Circumpolar Stars for the Epoch* 1810·0, by Frank Watson Dyson and William Grasett Thackeray, under the direction of Sir W. H. M. Christie. Practical astronomers speedily appreciated the value of the work. In a Council Note in *M.N.*, 1906 Feb., Kapteyn spoke of the proper motions as being arranged "in an extremely clear, concise, and complete way which will command the gratitude of any investigator of stellar motions," and in the same note he expressed the opinion that the volume "may safely be considered as the most valuable contribution of material for the study of sidereal motions since the publication of the third volume of Auwers-Bradley. In the same volume of the *M.N.* (1906 June), Professor Lewis Boss tested the accuracy of the re-reduction and spoke with high praise of the standard of precision attained.

One occasionally hears doubts expressed as to the wisdom of expending so vast an amount of labour on accumulating and reducing masses of observations of the type just considered. Such doubts do not come from the practical astronomers, whose whole life-work is a testimony to their faith in the value of observations, least of all do they come from the theoretical astronomers, whose whole life-work centres round a desire for more and more observations of ever higher and higher accuracy. They come usually, I think, from workers in other sciences, who often have a greater friendliness for astronomy than insight into its methods.

I do not know if there were any such doubters as to the value of our Medallist's work; if so, they were soon brought to confusion. The fates, to whom had been entrusted the weaving of astronomical history for the few years in question, had arranged for a very dramatic demonstration of the value both of Groombridge's observations and of the proper motions derived from them. In 1904 Kapteyn announced the existence of the two star streams with which his name will for ever be associated. His discovery was based only on the proper motions of about 2500 Bradley stars, which being all of sixth magnitude or brighter were, statistically speaking, stars quite near to the sun. Kapteyn's discovery was in actual fact that the stars in the immediate neighbourhood of the sun fell into two streams: no material was available in 1904 for discussion as to the motions of remote stars. Dyson's 1905 catalogue of proper motions at once provided the material needed for the discussion of star-streaming down to stars of magnitude 9·5, and in November 1906 Eddington was able to announce that these also conformed to Kapteyn's law of star-streaming. The existence of the Groombridge Catalogue in fact made it possible to search for star-streaming, and to find star-streaming, to distances about five times as remote as those discussed by Kapteyn. The question of whether star-streaming is merely a local phenomenon or extends through the whole of our galactic system is of the utmost importance; it cannot yet be fully answered, but the answer when it comes will probably bring with it the solution of the puzzle of the structure and origin of our system of stars. To obtain this answer we need ever more and more accurately measured proper motions extending to stars of greater and greater distance and faintness. Our Medallist made a contribution of notable value to this problem when, as Astronomer Royal for Scotland, he tested Kapteyn's hypothesis by means of the stars of large proper motion. The special advantage of using stars of large proper motion is, of course, that only a small fraction of the observed proper motion can arise from errors of observation.

It is now twenty years since our Medallist published his proper motions of the Groombridge stars; his subsequent work has always tended to centre round the field of his first activities. His return to Greenwich as Astronomer Royal in 1910 made it possible for him to take up the determination of proper motions in a more systematic and comprehensive way than was previously possible. He was no longer an almost solitary worker, but could call on the co-operation of the main part of the Greenwich staff. As a beginning, we find in vols. lxxiii. and lxxiv. of our *M.N.* (1913–1914) three noteworthy papers on the motions

of the stars in Carrington's catalogue of stars within 9° of the North Pole; one showing that 95 per cent. of the stars within the B.D. limit are more luminous than the sun, another dealing with the correlation between motion and spectral type, while the third showed that the ellipsoidal law of distribution which had been advanced by Schwarzschild fitted the observed facts rather better than the law obtained from the Kapteyn hypothesis of two intermingling streams of stars.

This last result was a valuable contribution towards the problem of determining the fundamental meaning of Kapteyn's star-streaming. The discoverer of the phenomenon had jumped at the most obvious interpretation which would fit the limited range of facts within his knowledge; he thought of the stars as two vast armies, meeting one another in space, and passing through one another without serious mutual interference. The more analytical mind of Schwarzschild saw that a more subtle, but in many ways more plausible, interpretation was open. To a first, and very rough, approximation we may think of the stars as analogous to the molecules of a gas, moving in all directions indiscriminately with velocities grouped about a mean. Kapteyn's suggested second approximation had compared their motions to those of the molecules of two separate masses of gas, intermingled for the moment, but each pursuing its course through space, and fated in time to reappear as two separate masses. Schwarzschild's second approximation compared the motion to that of the molecules of a single mass of gas, in which, for some reason unexplained, there was a " preferential motion " to and fro in one direction in space. Kapteyn's view regarded our stars as forming two separate systems, while Schwarzschild's view regarded them as forming essentially a single system. The result obtained by Dyson in the paper just mentioned gave considerable support to the view of Schwarzschild as against that of Kapteyn. At first sight the two hypotheses of Kapteyn and Schwarzschild seem to be in irreconcilable opposition, but I think it is possible that the distinction between the hypotheses is perhaps after all only one of degree. Armies of stars may get intermingled in the way imagined by Kapteyn, but instead of fighting their way through one another gravitational forces may provide attractions which result first in spasmodic fraternisations between members of the opposing armies, and finally in the two armies, now welded into one, marching through space as a single army, in which case it would exhibit precisely those peculiarities of motion which were suggested by Schwarzschild.

The year 1914 further saw the completion of the re-observation at Greenwich of the zone 24° N. to 32° N., and this provided material for the determination of the proper motions of some 12,000 stars. These proper motions were not merely tabulated but also were fully discussed in the Greenwich Catalogues and in various papers read before this Society (generally in collaboration with Thackeray), in which we find investigations of the mean parallaxes of stars of different magnitudes, the correlation between proper motions and galactic latitude, discussions of systematic motions, spectral type, the parallax of B-type stars, and similar topics.

Seven years later, vol. iv. of the Greenwich Section of the *Astro-*

graphic Catalogue was published. It contained the proper motions of some 12,000 stars between the pole and Dec. 64 N., including all those B.D. stars which show a sensible proper motion.

On the whole, we may say that Dyson has taken a very large share indeed in the determinations of the proper motion of faint stars (*i.e.* stars fainter than the Boss stars) north of 24° N. This share will be still further increased when the re-observation at Greenwich of all stars brighter than $8^{m}\cdot0$ between 32° N. and 64° N. is completed.

Such observations, systematic, complete in the sense of covering the whole of the field they set out to cover, perfect in the sense of being as refined as modern appliances and technique can make them, form the foundation on which astronomical progress is built. They are, in the last resort, the raw material of real advance in our knowledge of the universe in which we live. Or, to use a different metaphor, they are the solid substance out of which the main structure of the Temple of astronomical science is fashioned. Some may choose to erect picturesque but isolated little pinnacles on which nothing further can be built, but Dyson has always preferred to make a solid contribution to the main mass. He who elects to do this must be to some extent prepared to sink his personality in his science ; his reward must to a large extent be the knowledge that his work is valuable rather than the knowledge that the world appreciates its value, for the real value may not appear for generations ; Stephen Groombridge provides an instance ready to hand.

Our Medallist has in a sense gone even further than this ; he has sunk his personality in his observatory, so that his contributions to science almost appear to be those of his observatory. Of recent years all his papers have been published in collaboration with members of his staff ; it is in no way undervaluing the contributions of these collaborators to say that the guiding idea in the whole series of investigations has been that of the Astronomer Royal. There is always a danger in such cases that the leading spirit will keep the guiding idea to himself, while the collaborators sink to the level of mere observers, who regard the observations as an end in themselves and see nothing beyond. I venture to think our Medallist has been singularly successful in avoiding this danger. It is not generally possible for all the observers to have the wide vision of the director, but Dyson has, I know, endeavoured, and with success, to make each observer keep the object of his observations always in view. They have not been treated as mere hewers of stones, told to cut a stone to such and such a shape, but rather as co-architects, with whom the director has freely and frankly discussed his plans for the bit of the structure immediately in hand, and in consultation with whom he has also tried to envisage the whole edifice as it will finally appear. It is perhaps as much because he has taken this statesmanlike and generous view of his high position as because he has measured the proper motions of so many circumpolar stars, that the Society will feel the appropriateness of the Council's selection of their Medallist for this year.

I have barely left myself time to touch on his other scientific activities. I am told that the spectroscopists consider there is a debit

as well as a credit side to our Medallist's enthusiasm for proper motions ; it is that a first-class spectroscopist has been lost to science. They judge from his work on the eclipse expeditions of 1900, 1901, and 1905, which he accompanied as Greenwich Chief Assistant. His well-known paper in the *Phil. Trans.* for 1906 still provides the standard determination of coronal wave-lengths. In other regions, even where he has not taken an active part in the work himself, he has inspired his colleagues to do something more valuable than the mere recording of observations—I may refer particularly to the double-star observations, and to the way in which, although no doubt with Professor Chapman's enthusiastic concurrence, both old and current magnetic observations have been transformed from a mass of dead useless facts into a living body of scientific knowledge. Perhaps the supreme instance is, however, to be found in the *Astrographic Catalogue*. Too many of the Observatories which have taken part in this work have limited themselves, no doubt in many cases from scarcity of means or personnel, to the mere taking and measuring of plates which enabled them to say they had fulfilled their part of the contract ; the four Greenwich volumes so far issued exhibit, in ever-increasing measure, the process of transformation of bricks into an edifice.

I have kept for the last the instance which perhaps makes on me personally the greatest impression of all. In 1918, in the darkest days of the war, two expeditions were planned, one by Greenwich Observatory and one by Cambridge, to observe, if the state of civilisation should permit when the time came, the eclipse of May 1919, with a view to a crucial test of Einstein's generalised relativity. The armistice was signed in November 1918 ; the expeditions went, and returned bringing back news which changed, and that irrevocably, the astronomer's conception of the nature of gravitation and the ordinary man's conception of the nature of the universe in which he lives. If the credit of this achievement had to be divided between Sir Frank Dyson and Professor Eddington I frankly do not know in what proportion the division should be made. To my mind, however, it is not so much an occasion for sharing out credit as for attributing the whole credit to each, for if either had failed to play his part, either from want of vision, of enthusiasm, or of capacity for seizing the right moment, I doubt if the expeditions would have gone at all, and the great credit of first determining observationally what sort of things space and time really are would probably have gone elsewhere.

Sir Frank Dyson, it is a duty which falls to me as President to hand you this Medal ; it is a pleasure which I permit myself, and in which I am sure the whole Society will join, to offer you with it most cordial wishes for long-continued health and energy and for success in your future labours no less marked than that which has distinguished them in the past.

The Generation of Energy in a Star rotating as a Rigid Body. Reply to Dr Jeans' Criticism. By H. v. Zeipel, Assoc.R.A.S.

1. It is well known that the radiative equilibrium of a gaseous mass, whose level surfaces are plane or spherical, is undetermined. In the solution of these problems an arbitrary function enters. The generation of energy per unit mass per second within the gas may be an arbitrary function of its density and of its temperature. Thus there exists an infinity of spherical figures of equilibrium.

In a previous paper * I have treated a more general problem. I have studied the radiative equilibrium of a star rotating as a rigid body. It was shown that the solution of this more general problem is not incapable of determination. The rate of production of energy per unit mass, $4\pi\epsilon$, depends on density ρ by the simple formula

$$(1) \quad . \quad . \quad . \quad . \quad 4\pi\epsilon = B\left(1 - \frac{\omega^2}{2\pi G\rho}\right),$$

where ω is the angular velocity, G the constant of gravitation, and B a constant for the whole star depending on its mass and angular velocity.

I think, like Dr. Jeans,† that this theorem is very surprising. But I cannot share the opinion of Dr. Jeans that the formula (1) is erroneous.

In the paper cited ‡ I have explicitly excluded the special cases in which the intensity of gravity is constant on the level surface. Thus the exactly spherical stars were excluded. Also the artificial case of Schwarzschild, where the level surfaces are plane, must be excluded from the discussion.

In the first paragraphs (2–7) of his criticism, Dr. Jeans considers just the special case of Schwarzschild, in which my theorem is not valid. It seems unnecessary to enter here into this part of the criticism.

2. My analysis starts from Schwarzschild's equation

$$(2) \quad . \quad . \quad . \quad . \quad \frac{dI}{d\sigma} + I = \mu T^4 \equiv E,$$

where I is the intensity of radiation at the point x, y, z, and in the direction α, β, γ. σ is the optical mass in this direction. Thus

$$(3) \quad . \quad . \quad . \quad \frac{1}{d\sigma} = \frac{1}{k\rho ds} = \frac{1}{k\rho}\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}\right),$$

where ρ is the density and k the absorption-coefficient. Considering, on the right side of (2), the temperature T as a known function of σ, the solution satisfying the condition that the ingoing intensity is zero ($I=0$ for $\sigma=0$) is

$$(4) \quad . \quad . \quad . \quad . \quad I(\sigma) = \int_0^\sigma e^{\sigma'-\sigma} E(\sigma') d\sigma'.$$

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* *M.N.*, **84**, 665. † *M.N.*, **85**, 526. ‡ *Loc. cit.*, p. 674.

After three or four partial integrations this solution may be written in the form

$$\mathrm{I}=\mathrm{E}-\frac{d\mathrm{E}}{d\sigma}+\frac{d^2\mathrm{E}}{d\sigma^2}+\mathrm{I}_3$$

$$=\mathrm{E}-\frac{d\mathrm{E}}{d\sigma}+\frac{d^2\mathrm{E}}{d\sigma^2}-\frac{d^3\mathrm{E}}{d\sigma^3}+\mathrm{I}_4. \quad (5)$$

The expressions of the remainder terms I_3 and I_4 appear from the unnumbered formula between (12) and (13) in my article, *loc. cit.*, **84**, 669. The expressions (4) and (5) are perfectly exact.

The infinite series

$$\mathrm{E}-\frac{d\mathrm{E}}{d\sigma}+\frac{d^2\mathrm{E}}{d\sigma^2}-\ldots \quad (6)$$

is evidently not convergent for every function of E. However, the optical mass σ is (when far enough from the star's surface) a very large number, comparable with

$$\sigma_1=k\bar{\rho}\mathrm{R}\sim 10^{10}, \quad (7)$$

where $\bar{\rho}$ is the mean density and R the mean radius of the star. Consequently the series (6) is semi-convergent. The five terms in the right-hand side of (5) are very rapidly decreasing in a ratio $\sim\sigma_1^{-1}$.

On account of (3) the four first terms in the right-hand side of (5) are homogeneous polynomials of α, β, γ of the degrees 0, 1, 2, 3. The remaining term I_4 is a more complicated function of α, β, γ.

3. The rate of generation of energy per unit mass is equal to the difference between the emission and the absorption per unit mass per second. Thus

$$4\pi\epsilon=k[4\pi\mathrm{E}-\int \mathrm{I}d\Sigma],$$

where $d\Sigma$ is the elementary solid angle in the direction α, β, γ. Introducing in the integral the expression (5) of I, we find

$$\int \mathrm{E}d\Sigma=4\pi\mathrm{E}, \quad \int\frac{d\mathrm{E}}{d\sigma}d\Sigma=0, \quad \int\frac{d^3\mathrm{E}}{d\sigma^3}d\Sigma=0.$$

Thus

$$4\pi\epsilon=-k\left[\int\frac{d^2\mathrm{E}}{d\sigma^2}d\Sigma+\int \mathrm{I}_4 d\Sigma\right]$$

$$=-k\left[\frac{4\pi}{3}\sum_3\frac{d}{k\rho dx}\left(\frac{d\mathrm{E}}{k\rho dx}\right)+\int \mathrm{I}_4 d\Sigma\right], \quad (8)$$

where in the sum $\sum_3$ of three terms the second and third terms are obtained from the first by writing y and z instead of x.

Neglecting the remainder term of (8) means neglecting $\sigma_1^{-2}\sim 10^{-20}$. Thus we have with very close approximation the expression

$$\epsilon=-\frac{1}{3\rho}\sum_3\frac{d}{dx}\left(\frac{d\mathrm{E}}{k\rho dx}\right)+\ldots \quad (9)$$

valid except in the immediate vicinity of the star's surface.

Making this approximation I have not supposed (as Dr. Jeans considers, *loc. cit.*, **84**, 529) that all the derivatives

$$\frac{d^3E}{d\sigma^3}, \quad \frac{d^4E}{d\sigma^4}, \quad \ldots$$

are zero, nor that the series (6) is convergent. These derivatives may increase indefinitely with increasing order. I have only supposed that the remainder term in (8) is small in comparison with the principal term. Evidently this is the case when the quantity σ_1 in (7) is large enough, supposing only, *e.g.*, that the stars of given oblateness are roughly similar or homologous in respect to form, density and temperature distribution.

4. Assuming as a fundamental hypothesis that the nature of the gas is constant over the level surface (which is always the case when the star consists of a single gas) I have shown, *loc. cit.*, **84**, 672, that T, ρ, k are functions of the gravity potential Φ only. Then the formula (9) may be put in the form

$$\epsilon=-\frac{1}{3\rho}\left[\frac{dE}{k\rho d\Phi}\Delta\Phi+\frac{d}{d\Phi}\left(\frac{dE}{k\rho d\Phi}\right)\left(\frac{d\Phi}{dn}\right)^2\right]+\ldots, \tag{10}$$

where $\frac{d\Phi}{dn}$ is the intensity of gravity.

Introducing the notations

$$f=\frac{dE}{k\rho d\Phi}, \quad F=\frac{\left(\frac{d\Phi}{dn}\right)^2}{4\pi G\bar{\rho}\Phi},$$

and remembering that

$$\Delta\Phi=-4\pi G\rho\left[1-\frac{\omega^2}{2\pi G\rho}\right],$$

the formula (10) may be written more explicitly

$$\epsilon=\tfrac{4}{3}\pi Gf\left[1-\frac{\omega^2}{2\pi G\rho}-\frac{d\log f\,\bar{\rho}}{d\log\Phi\,\rho}\cdot F\right]+\ldots. \tag{11}$$

The function f is >0, while T and Φ are increasing together. All the functions written explicitly in (11), except F, are constant on the level surface.

I have shown * that the intensity of gravity varies on the level surface, if $\omega>0$. It is well known, *e.g.* from the theory of Maclaurin's ellipsoids or from geodesy, that the relative variations of gravity along a level surface are of the order of the oblateness. This is certainly the case also for gaseous masses in equilibrium.

If M is the mass and R the mean radius of the star, we have for moderate oblateness

$$\Phi\sim\frac{GM}{R}, \quad \frac{d\Phi}{dn}\sim\frac{GM}{R^2}, \quad \bar{\rho}\sim\frac{M}{R^3}.$$

* *M.N.*, **84**, 680.

Thus $F \sim 1$, or more precisely

$$F \sim \bar{F}(1+\delta).$$

The mean value $\bar{F}$ on the level surface is ~ 1. The variable quantity δ is comparable with the oblateness.

Dr. Jeans considers that the variable part of the expression (11), viz. the term in δ, is in general comparable with the neglected remainder term. However, I am not of the same opinion. In reality the term depending on δ is

$$\sim G f \frac{d \log f}{d \log \Phi} \frac{\bar{\rho}}{\rho} o,$$

where o indicates the oblateness. On the other hand, the remainder term is

$$\sim k E \sigma_1^{-4}.$$

Thus in order that ϵ may be constant on the level surface the function f cannot be arbitrary but must satisfy the condition

$$\frac{d \log f}{d \log \Phi} \sim k E \sigma_1^{-4} G^{-1} f^{-1} o^{-1} \rho \bar{\rho}^{1-}$$
$$\sim k^2 E \sigma_1^{-4} G^{-1} \frac{d\Phi}{dE} o^{-1} \rho^2 \bar{\rho}^{-1} \sim k^2 \sigma_1^{-4} \Phi G^{-1} \frac{d \log \Phi}{d \log E} o^{-1} \rho^2 \bar{\rho}^{-1}.$$

But on account of the relations

$$\Phi G^{-1} \sim M R^{-1} \sim \bar{\rho} R^2$$
$$k^2 \sigma_1^{-2} \sim \bar{\rho}^{-2} R^{-2}$$

we have

$$k^2 \sigma_1^{-2} \Phi G^{-1} \sim \bar{\rho}^{-1}.$$

Thus the condition above can be written

$$(12) \quad . \quad . \quad \frac{d \log f}{d \log \Phi} \sim \frac{d \log \Phi}{d \log E} \left(\frac{\rho}{\bar{\rho}}\right)^2 o^{-1} \sigma_1^{-2} \sim o^{-1} \sigma_1^{-2},$$

excluding the immediate vicinity of the surface.

This condition proves that the function f must be practically constant in the interior of a star rotating as a rigid body, supposing only that the oblateness is not exceedingly small.

Accordingly the expressions (11) and (1) are practically identical.

I hope that the more detailed exposition of the theorem (1) given above will disperse the doubts of Dr. Jeans.

On the Effects of Long Exposures in Astronomical Photography.

By W. M. Smart, M.A., D.Sc.

(*Communicated by the Director, Cambridge University.*)

§ 1. Many recent researches on stellar positions have been based on photographs exposed for several hours, and it seems pertinent to inquire into the possible effects on the stellar images, imprinted on the plates, due to such exposures. In this paper, accurate guiding is assumed.

The principal effect is due to refraction. During a long exposure, the zenith distances of the stars in the field are continually altering, and the rate of variation is sensibly different for different positions on the plate. There is thus a differential effect depending on the coordinates of the star considered. The result is, in general, a distortion of the images. Some years ago, Mr. A. R. Hinks, in a paper "On some Attempts to Counteract certain Effects of Refraction" (*M.N.*, **58**, 428), made some calculations on the rate of distortion of stellar images, due to refraction; these were, however, incidental to the chief object of the paper, viz. a discussion of possible instrumental adjustments which would tend to counteract, in practice, the effects of refraction.

To obtain numerical estimates of the various effects produced, I have considered, in some detail, in the latter part of the paper, a simple case: the centre of the region photographed is in declination 0° and the latitude of the observatory is assumed to be 45°.

§ 2. In fig. 1, C is the centre of the plate, and Q and W are the projections on the plate of the pole P and the zenith Z respectively. T is the projection on the plate of a star S undisplaced by refraction. The instantaneous centres of the refracted images of stars at C and S will be at C_0 and T′. If β is the coefficient of refraction and z is the zenith distance of C, the displacement CC_0, denoted by d_0, is given by

$$d_0=\beta \tan z.$$

The angle QCW, denoted by l, is the same as the parallactic angle PCZ on the sphere. The displacement TT′ is $d_0+\Delta d$. If we suppose that the guiding star is at C, then we need only consider the motion of T′, during a long exposure, relative to T_0, where TT_0 is drawn equal and parallel to CC_0. The coordinates of T′ with respect to T_0 are $\Delta\xi$, $\Delta\eta$, and these are given by the well-known formulæ

$$\Delta\xi=-\beta(1+X^2)\xi-\beta XY\eta,$$
$$\Delta\eta=-\beta XY\xi-\beta(1+Y^2)\eta,$$

in which X, Y are the plate coordinates of W.

We write these in the form:—

$$\Delta\xi=-a\xi-b\eta \quad . \quad . \quad . \quad . \quad (1)$$

$$\Delta\eta=-b\xi-c\eta \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\left.\begin{aligned} a&=\beta(\tan^2 z \sin^2 l+1)\\ b&=\beta \sin l \cos l \tan^2 z\\ c&=\beta(\tan^2 z \cos^2 l+1)\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

a, b, c are thus the plate constants so far as refraction is concerned.

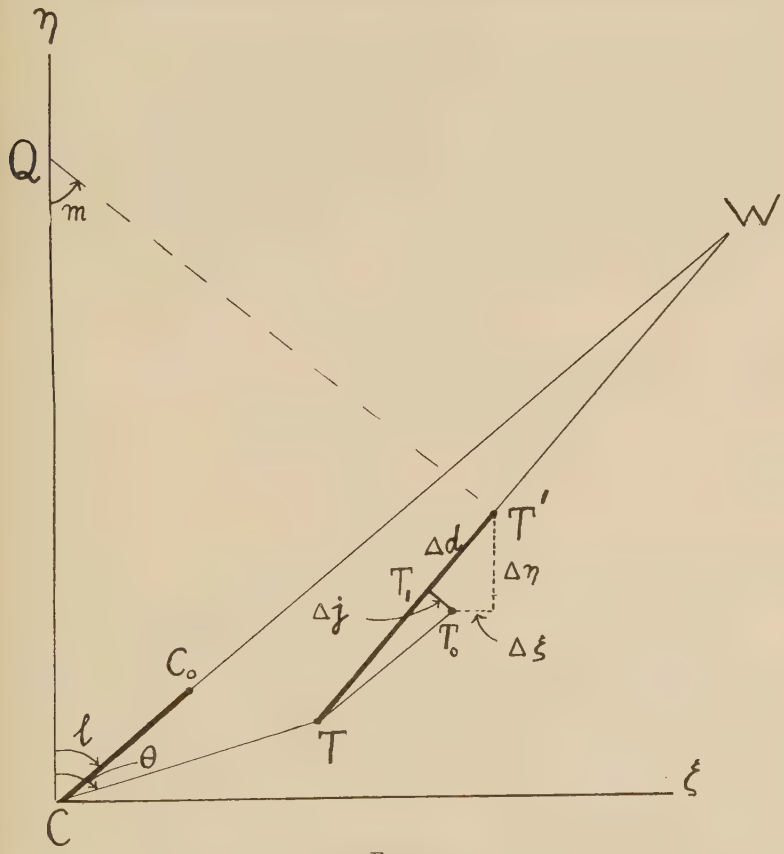

Fig. 1.

The position of T′ relative to T_0, at any moment, is also given by the quantities Δd and Δj. The values of Δd and Δj are found from the formulæ

$$\Delta d=\Delta\xi \sin l+\Delta\eta \cos l,$$
$$\Delta j=-\Delta\xi \cos l+\Delta\eta \sin l.$$

In particular, I consider a ring of stellar images distant ϕ from the centre of the plate and in position angles θ. We have thus

$$\xi=\phi \sin\theta \quad \eta=\phi \cos\theta,$$

and the values of Δd and Δj are easily obtained in the form

$$\Delta d = -\beta\phi \sec^2 z \cos(\theta - l) \quad . \quad . \quad . \quad (4)$$

and

$$\Delta j = \beta\phi \sin(\theta - l) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For large zenith distances, Δd gives the major part of the displacement T_0T'.

§ 3. From (1) and (2), it is seen that $\Delta\xi$, $\Delta\eta$ are functions of the time in terms of which z and l can be easily expressed. Thus the point T′ traces out a path relative to T_0, during a long exposure, so that the integrated image must be supposed to be built up of a series of instantaneous impressions on the plate, the centres of the impressions lying on a curve found by eliminating the time (or hour angle) in (1) and (2). The image of a star will then be distorted in a definite and calculable manner and the character of the distortion will vary according to the value of θ.

To obtain the equation of the curve, write x, y for $\Delta\xi$ and $\Delta\eta$ so that

$$\left.\begin{aligned} x \sin l + y \cos l &= \Delta d \\ x \cos l - y \sin l &= -\Delta j \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

From these, using (4) and (5), we derive

$$(tx+y)(1-Ct) = \sec^2 z\,(C+t)(x-ty) \quad . \quad . \quad . \quad (7)$$

where

$$t = \tan l$$

and

$$C = \cot\theta,$$

and sec z is expressed in terms of t by the equation

$$(1+t^2)(\sec z \cos c - \sin\delta) = \cos^2\delta(\sec^2\delta - 1),$$

in which c is the colatitude of the observatory and δ is the declination of the centre of the field photographed.

From the second equation of (6)

$$t(y+\beta\cos\theta) = x + \beta\phi\sin\theta \quad . \quad . \quad . \quad (8)$$

and the equation of the trail of T′ is found finally by eliminating t between this equation and (7).

The curve is, in general, of a complex nature.

In the particular case—$c = 45°$ and $\delta = 0°$—the equation (7) reduces to

$$y - 2Cx + t(Cy - x) + t^2(y - Cx) - xt^3 + Cxt^4 = 0,$$

and it is readily seen from this equation and (8) that the curve has two asymptotes parallel to $y = \pm x$.

When $\theta = 90°$, the curve is simply the rectangular hyperbola

$$y^2 = x^2 + \beta\phi x,$$

the left-hand branch of the curve being alone applicable, as may be verified from Table II. below. The nature of the asymptotes may also

be inferred from the values of Δd and l in Table I., for the maximum value of l is 45° (corresponding to zenith distance 90°), and the values of Δd increase without limit as the position is approached. This is not a point of practical interest, for it is assumed that the refraction varies as tan z, whereas it is finite for all zenith distances, including 90°.

For large zenith distances, the value of Δd is predominantly large as compared with Δj, and the general curve is approximately given, under these circumstances, by

$$y^4-x^4+2\beta\phi y^2(x \sin\theta+y\cos\theta)=0.$$

The form of the curve for typical points on the plate will be derived below.

§ 4. During a long exposure, the values of a, b, and c—the refraction-plate constants—are continually altering. To find the rates, we have, denoting the hour angle and azimuth at any moment by H and A respectively,

$$\cos z=\sin\delta\cos c+\cos\delta\sin c\cos \mathrm{H},$$

so that

$$\dot{z}=\cos\delta\sin l\frac{d\mathrm{H}}{dt}$$

or

$$\dot{z}=\sin c\sin \mathrm{A}\frac{d\mathrm{H}}{dt}.$$

Again,

$$\cos c=\sin\delta\cos z+\cos\delta\sin z\cos l,$$

so that

$$\dot{l}=-\frac{\cot \mathrm{A}}{\sin z}\cdot\dot{z}.$$

From these, the value of $\dot{a}$ is given by

$$\dot{a}=2\beta\sin l\cos c\tan z\sec^2 z(\cos\delta\sin c+\sin\delta\cos \mathrm{H}\cos c)\frac{d\mathrm{H}}{dt}.$$

Also

$$a+c=\beta(\sec^2 z+1),$$

and we find that

$$\dot{c}=2\beta\cos l\sin c\cos c\sin \mathrm{H}\tan z\sec^2 z\frac{d\mathrm{H}}{dt}.$$

The expression for $\dot{b}$ is :

$$\dot{b}=\beta\sin c\tan z\sec^2 z(\sin 2l\sin \mathrm{A}-\cos 2l\cos \mathrm{A}\cos z)\frac{d\mathrm{H}}{dt}.$$

For an equatorial region, the expressions are much simpler. They are :

$$\dot{a}=2\beta\tan \mathrm{H}\sec^2 \mathrm{H}\frac{d\mathrm{H}}{dt},$$

$$\dot{b}=\beta\sec \mathrm{H}(\sec^2 \mathrm{H}+\tan^2 \mathrm{H})\frac{d\mathrm{H}}{dt},$$

$$\dot{c}=2\beta\cot^2 c\tan \mathrm{H}\sec^2 \mathrm{H}\frac{d\mathrm{H}}{dt}.$$

At some distance from the meridian, these rates are considerable. In particular, if we take the case of latitude 45° and declination 0° and an exposure from hour angle 0^h to hour angle 3^h, we find that the values of a, b, and c at the beginning and at the end of the exposure are given by:

H.A. 0^h: $a=\beta$; $b=0$; $c=2\beta$.
H.A. 3^h: $a=2\beta$; $b=\frac{4}{5}\beta$; $c=3\beta$.

It is easily seen that the greater part of the changes in a, b, and c are due to the greater rates of change of these quantities in the later parts of the exposure.

During this time, the zenith distance has increased from 45° to 60°.

§ 5. If u and v denote the displacements of the central image at C_0 from its standard position at C, we have

$$u=d_0 \sin l \text{ and } v=d_0 \cos l$$

or

$$u=\beta \tan z \sin l,$$

and

$$v=\beta \tan z \cos l.$$

If the central star is the guiding star, the rates of guiding are $\dot{u}$ in right ascension and $\dot{v}$ in declination. We derive

$$\dot{u}=\beta \sin c \sec^2 z (\sin l \sin \mathrm{A}-\cos l \cos z \cos \mathrm{A})\frac{d\mathrm{H}}{dt},$$

which can be put in the form

$$\dot{u}=\beta \sin c \sec^2 z (\sin c \cos \delta+\cos c \sin \delta \cos \mathrm{H})\frac{d\mathrm{H}}{dt},$$

and also

$$\dot{v}=\beta \sin c \cos c \sec^2 z \sin \mathrm{H}\frac{d\mathrm{H}}{dt}.$$

For an equatorial region and latitude 45°, the guiding rates are:

in R.A.: $0''{\cdot}25 \sec^2 \mathrm{H}$ per sidereal minute,
in declination: $0''{\cdot}25 \sin \mathrm{H} \sec^2 \mathrm{H}$ per sidereal minute,

of which the former is, clearly, the more important.

§ 6. During a long exposure, there is a further effect, namely, the rotation of the field—as was pointed out by Mr. Hinks in *M.N.*, **58**, 434. In fig. 1, let m denote the angle CQT′. Suppose T′ is the image of the guiding star. Now TQ makes a constant angle with the η-axis, but as the refracted image at T′ is constantly altering with regard to the standard axes, the angle m consequently is constantly altering. The effect of guiding on the image at T′ is thus to produce a retrograde rotation of the η-axis. The result is a rotation of the field with angular velocity $\dot{m}$.

If d denotes the displacement TT′ and ϵ the angle CTT′, and a denotes CQ, that is, cot δ, we obtain easily for a non-polar region

$$\tan m=\frac{\xi}{a-\eta}-\frac{d}{a}\sin(\theta+\epsilon),$$

in which only the first order terms in ξ, η, and d have been retained. The abandonment of higher order terms is equivalent—as regards the value of $\dot{m}$—to regarding the central star as the guiding star. Also, we have, with sufficient accuracy for the present purpose,

$$\theta+\epsilon=180^\circ+l.$$

m is small for a non-polar region, and we obtain simply

$$a\dot{m}=\dot{d}\sin l+d\cos l\,.\,\dot{l}.$$

The important part of d is $\beta\tan z$. Using the expressions for $\dot{z}$ and $\dot{l}$ previously obtained, we find that the rotation $\dot{m}$ is given by the formula

$$\dot{m}=\beta\tan\delta\sin c\sec^2 z\,(\sin c\cos\delta+\cos c\sin\delta\cos \mathrm{H})\frac{d\mathrm{H}}{dt}.$$

The guiding rate in right ascension is, for the central star, $\dot{u}$ (see § 5), so that

$$\dot{m}=\dot{u}\tan\delta.$$

The rotation vanishes for equatorial regions and is only of consequence in high declinations.

If the centre of the region photographed is in the neighbourhood of the pole, we have approximately

$$l=180^\circ-\mathrm{H},$$

so that

$$\theta+\epsilon=360^\circ-\mathrm{H}.$$

Since d is a small quantity, the formula for m becomes

$$\tan m=\frac{\xi}{a-\eta}+\frac{1}{a-\eta}\left(d\sin \mathrm{H}-\frac{\xi d}{a-\eta}\cos \mathrm{H}\right).$$

In this equation it is assumed that $a-\eta>d$, that is to say—for practical purposes—that the guiding star is taken at some little distance from the pole.

Now, for polar regions, the zenith distance throughout a long exposure will undergo comparatively small variations, so that d, in the preceding formula, may be regarded as sensibly constant. This formula immediately gives the measure of the rotation. If the guiding star is at C or on the meridian of C, we have simply

$$\dot{m}=\frac{\beta}{p}\tan z\cos \mathrm{H}\frac{d\mathrm{H}}{dt},$$

in which p is the polar distance of the guiding star. As remarked already, the zenith distance z of C will remain fairly constant during an exposure, so that the rotation of the field will be a maximum for plates exposed on the meridian and will disappear at hour angles 6^h or 18^h. For example, if $p=1^\circ$ and the latitude is 45°, the rotation of the field is given approximately by

$$\dot{m}=14''\cdot6\cos \mathrm{H}$$

per sidereal minute.

§ 7. I consider in greater detail the effects of a long exposure made in latitude 45°, the declination of the field being 0°. In the second column of Table I. are found the values of the zenith distance at hour

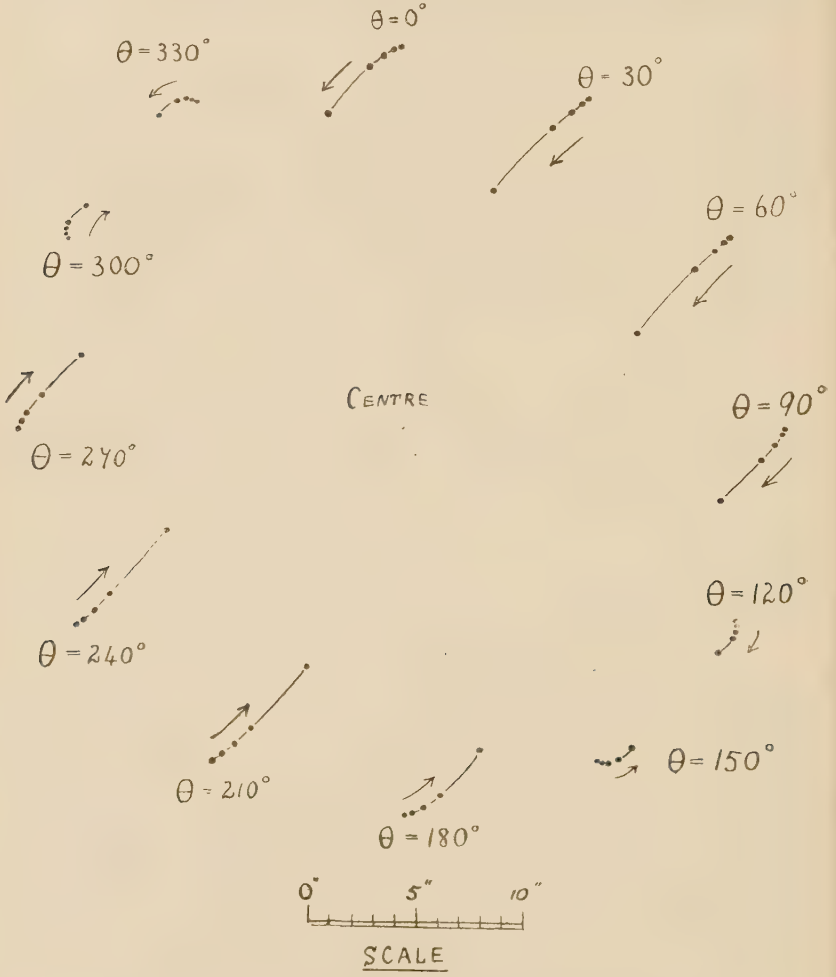

Fig. 2.

angle intervals of 20^m from 0^h to 4^h—these have been taken directly from Ball's *Altitude Tables*. Since the declination is 0°, we have

$$\tan l = \sin H \tan c.$$

The values of l have been taken directly from *Position Line Tables* (Potter) by W. M. Smart and F. N. Shearme, where angles derived from precisely the same formula are tabulated ; these are given in the third

column. I have considered 12 positions of a stellar image, distant 1° from the centre of the plate and in position angles 0°, 30°, . . . 330°, so that formulæ (4) and (5) become

$$\Delta d = -1''\cdot 815 \sec^2 z \cos(\theta - l),$$
$$\Delta j = 1''\cdot 015 \sin(\theta - l).$$

The remainder of the table gives the values of Δd and Δj—the first is Δd and Δj is underneath—for the six values of θ, 0°, 30°, . . . 150°. The values for the remaining values of θ are the same in magnitude but opposite in sign as compared with the diametrically opposite positions. If we wish to consider a ring of stellar images distant 20′ from the centre, we require to divide the tabulated quantities by 3—and so on.

TABLE I.

Values of z, l, and $\frac{\Delta d}{\Delta j}$.

Hour Angle.	Zenith Distance z.	l.	θ. 0°.	30°.	60°.	90°.	120°.	150°.
h m	°	°	″	″	″	″	″	″
0 00	45·0	0·0	−2·03	−1·76	−1·01	0·00	+1·01	+1·76
			0·00	+0·51	+0·88	+1·02	+0·88	+0·51
20	45·2	5·0	−2·04	−1·86	−1·17	−0·18	+0·87	+1·68
			−0·09	+0·43	+0·84	+1·01	+0·92	+0·58
40	45·9	9·9	−2·06	−1·96	−1·34	−0·36	+0·72	+1·60
			−0·18	+0·35	+0·78	+1·00	+0·96	+0·66
1 00	46·9	14·5	−2·10	−2·09	−1·52	−0·54	+0·58	+1·55
			−0·25	+0·27	+0·73	+0·99	+0·98	+0·71
20	48·4	18·9	−2·18	−2·26	−1·73	−0·75	+0·44	+1·51
			−0·33	+0·20	+0·67	+0·96	+1·00	+0·77
40	50·1	22·9	−2·27	−2·45	−1·97	−0·98	+0·30	+1·49
			−0·40	+0·12	+0·61	+0·94	+1·01	+0·81
2 00	52·2	26·6	−2·42	−2·70	−2·26	−1·21	+0·15	+1·49
			−0·45	+0·06	+0·56	+0·91	+1·01	+0·85
20	54·9	29·8	−2·62	−3·02	−2·61	−1·50	0·00	+1·52
			−0·51	+0·00	+0·51	+0·88	+1·01	+0·88
40	57·2	32·7	−2·91	−3·46	−3·07	−1·87	−0·16	+1·58
			−0·55	−0·05	+0·47	+0·86	+1·01	+0·91
3 00	60·0	35·3	−3·31	−4·04	−3·68	−2·35	−0·37	+1·70
			−0·59	−0·09	+0·42	+0·83	+1·01	+0·93
20	63·0	37·5	−3·89	−4·87	−4·53	−2·99	−0·64	+1·88
			−0·62	−0·13	+0·39	+0·81	+1·01	+0·94
40	66·1	39·3	−4·76	−6·08	−5·76	−3·90	−1·02	+2·17
			−0·64	−0·16	+0·36	+0·79	+1·00	+0·95
4 00	69·3	40·9	−6·14	−7·97	−7·67	−5·32	−1·55	+2·64
			−0·66	−0·19	+0·33	+0·77	+1·00	+0·96

In Table II. are given the corresponding values of $\Delta\xi$ and $\Delta\eta$ (the latter underneath the former) for the six values of θ, 0°, 30° . . . 150°. The displacements have the opposite sign in the other semicircle. From these the trace of the geometrical centre of the instantaneous image is obtained, on a large scale, as shown in fig. 2. The image centres at 0^h, 1^h, 2^h, 3^h, and 4^h are shown by dots, and the arrows are in the sense : 0^h to 4^h. It is to be remembered that in these results the guiding star is supposed to be at the centre. If it is one of the ring of stars at θ_1, the appropriate displacements are obtained by applying, with reversed sign, the tabulated values for θ_1 to all the other values given for the various selected positions of the stellar images.

TABLE II.

Values of $\frac{\Delta\xi}{\Delta\eta}$.

Hour Angle.	0°.	30°.	60°.	90°.	120°.	150°.
h m						
0 00	0″·00	−0″·51	−0″·88	−1″·02	−0″·88	−0″·51
	−2·03	−1·76	−1·01	0·00	+1·01	+1·76
20	−0·09	−0·59	−0·94	−1·03	−0·84	−0·43
	−2·04	−1·81	−1·10	−0·09	+0·95	+1·72
40	−0·18	−0·68	−1·00	−1·05	−0·82	−0·37
	−2·06	−1·87	−1·19	−0·18	+0·88	+1·69
1 00	−0·28	−0·78	−1·09	−1·10	−0·80	−0·30
	−2·10	−1·98	−1·29	−0·27	+0·80	+1·68
20	−0·40	−0·92	−1·19	−1·15	−0·80	−0·24
	−2·17	−2·08	−1·42	−0·41	+0·74	+1·68
40	−0·53	−1·06	−1·33	−1·25	−0·81	−0·17
	−2·25	−2·20	−1·57	−0·53	+0·67	+1·69
2 00	−0·68	−1·25	−1·61	−1·35	−0·83	−0·09
	−2·36	−2·38	−1·77	−0·67	+0·58	+1·71
20	−0·87	−1·51	−1·74	−1·51	−0·87	+0·01
	−2·52	−2·62	−2·01	−0·86	+0·50	+1·75
40	−1·11	−1·83	−2·06	−1·74	−0·93	+0·08
	−2·75	−2·94	−2·33	−1·11	+0·41	+1·82
3 00	−1·44	−2·26	−2·47	−2·02	−1·05	+0·22
	−3·03	−3·35	−2·76	−1·45	+0·28	+1·93
20	−1·88	−2·86	−3·06	−2·46	−1·20	+0·39
	−3·46	−3·94	−3·35	−1·86	+0·10	+2·06
40	−2·52	−3·75	−3·92	−3·08	−1·43	+0·63
	−4·09	−4·81	−4·23	−2·52	−0·16	+2·28
4 00	−3·53	−5·09	−5·28	−4·02	−1·78	+1·00
	−5·06	−6·13	−5·57	−3·51	−0·52	+2·62

The calculations have been taken further than would be consistent with practice—the range in zenith distance corresponding to a hypothetical exposure from hour angle 0^h to 4^h is $45°$ to $70°$. The main point is that the effect as exhibited in fig. 2 increases very rapidly with the hour angle. To demonstrate this more clearly, I have indicated by dots the instantaneous image centre at 0^h, 1^h, 2^h, 3^h, and 4^h for each of the selected stars in the figure. Actually, the scale of the curves shows a magnification of 195 as compared with the dimensions of the field.

Generally, the result of a long exposure will be the distortion or elongation of a stellar image in the direction given by the curves of fig. 2. For an exposure from 0^h to 1^h the effect will hardly be capable of detection, and consequently the image will be sensibly circular. But for larger hour angles the distortion will not be negligible, and the measures of the stellar images will be affected in consequence.

A further effect on the form of the stellar images may be briefly noticed, namely, that due to atmospheric dispersion. If a star in the ring considered is of different spectral type from that of the guiding star, a quantity proportional to $\tan z$ must be supposed to be incorporated in the values of Δd. This correction, however, is small in magnitude as compared with the values of Δd given in Table I., for the latter is proportional to $\sec^2 z$, and at large zenith distances is all-important.

§ 8. The displacements of the instantaneous image-centre throughout an exposure cannot be interpreted even in part as a rotation about the centre C of the field. If ϕ, θ are the polar coordinates with respect to C of a point on the curve, and $\phi+\Delta\phi$, $\theta+\Delta\theta$ those of an adjacent point, we have

$$\phi \,.\, \Delta\theta = -\Delta d \sin(\theta - l) - \Delta j \cos(\theta - l),$$

so that from (4) and (5)

$$\Delta\theta = \tfrac{1}{2}\beta \tan^2 z \sin(2\theta - 2l).$$

In our special case, $c=45°$, $\delta=0°$, we obtain

$$\Delta\theta = \tfrac{1}{2}\beta(\sin 2\theta - 2\cos 2\theta \tan \mathrm{H} \sec \mathrm{H}),$$

so that

$$\frac{d}{dt}(\Delta\theta) = -\beta \cos^2\theta \sec \mathrm{H} (\sec^2 \mathrm{H} + \tan^2 \mathrm{H}) \frac{d\mathrm{H}}{dt}.$$

This is a function of the position of the image on the plate.

§ 9. The results of the preceding paragraphs would seem to imply the existence of a magnitude equation of a systematic kind in the comparison of two plates of the same region taken at different hour angles. Let us suppose that one plate has been exposed between hour angles 23^h and 1^h and the other between 1^h and 3^h. Assume, for simplicity, that the stars in the ring are taken as comparison stars and that they are all of the same magnitude. Suppose further that the object of the research is the investigation of the motion of faint objects which can just be photographed during this exposure. For example, let us take the interval between the plates to be 10 years. Now in the first plate,

the images will be almost perfectly circular and their centres will be practically at the points given by an instantaneous exposure on the meridian. In the second plate, the images of the stars will be sensibly elongated but the apparent displacement of their geometrical centres will be eliminated in the reduction, for the plate constants a, b, and c (so far as refraction is concerned) will be the mean of their instantaneous values throughout the exposure. But owing to atmospheric absorption the most effective part of the second exposure for the extremely faint objects will be, say, from 1^h to 2^h, and the tendency in measuring these will be to direct the attention to the integrated images from 1^h to 2^h. The result will be an apparent displacement of the faint object in the direction opposite to that in which a brighter comparison star, supposed in the same position angle as the faint object measured, is displaced.

Assuming that these qualitative considerations are sustained, it is possible to obtain a rough estimate of this apparent and fictitious motion of the faint object. Consider two plates, the exposures of which are from 23^h to 1^h and 1^h to 3^h respectively. The mean values of $\Delta\xi$ and $\Delta\eta$ for a star 1° from the centre and in position angle 0° will be

$$0''{\cdot}0 \text{ and } -2''{\cdot}0 \text{ for the first exposure,}$$

and

$$-0''{\cdot}7 \text{ and } -2''{\cdot}4 \text{ for the second exposure.}$$

For a faint object in the neighbourhood of the star, the corresponding values of $\Delta\xi$ and $\Delta\eta$ will be

$$0''{\cdot}0 \text{ and } -2''{\cdot}0 \text{ for the first exposure,}$$

and

$$-0''{\cdot}5 \text{ and } -2''{\cdot}2 \text{ for the second exposure,}$$

the assumption being made that the effective image of the faint object in the second exposure is that from 1^h to 2^h. Relatively to the star, the displacement of the faint object is $0''{\cdot}2$ in each coordinate, which would indicate a fictitious proper motion of $2''$ per century in each coordinate, with our time interval of 10 years.

This numerical estimate does not pretend to indicate more than the order of magnitude of the effect.

Similar results follow when the guiding star is not at the centre.

It is clear that the question of magnitude equation, in this sense, does not arise when the mid-exposure of both plates is at hour angle 0^h.

Observatory, Cambridge:
1925 *May* 6.

A Modification of Gauss's Method for the Determination of Orbits.

By G. Merton, M.A.

Innumerable memoirs and papers have been published on the subject of the determination of the orbits of comets and minor planets, and no year passes without some addition to their number. It seems right, therefore, in adding to their list, to indicate at the outset what practical advantages are claimed in the method to be described. Experience with it has shown it to be very convenient for the computation of a preliminary orbit, and it is thought that other computers too may find it not only more convenient than other published methods, but leading also more directly to a satisfactory result, whether for a general orbit or a parabola. The classical methods appear unnecessarily complicated to the student, but the free use of rectangular co-ordinates, which is here made, renders the whole process very easy to follow. Although the method can be conveniently worked by anyone accustomed to the use of addition and subtraction logarithms, yet the full advantages will probably be appreciated mostly by the growing numbers of computers who use calculating machines.* In this connection must be mentioned a memoir by C. Veithen, *Über die Verwendung der Rechenmaschine bei der Bahnbestimmung von Planeten* (Leipzig, 1912), which only came to my notice when the preparation of the present paper had been commenced. The method he developed for the general orbit is similar to that to be described.

The General Orbit.—In the determination of an orbit there are two conditions which must be fulfilled in order to solve the problem, namely the geometrical and the dynamical conditions. We commence with the equations in the heliocentric rectangular co-ordinates (x, y, z) of the body:

$$(1) \quad \left\{\begin{aligned} c_1x_1 - x_2 + c_3x_3 &= 0 \\ c_1y_1 - y_2 + c_3y_3 &= 0 \\ c_1z_1 - z_2 + c_3z_3 &= 0. \end{aligned}\right.$$

These include the geometrical condition that the plane of the orbit shall pass through the sun. If, moreover, the values of c_1 and c_3, which are the ratios of the areas of the *triangles* described about the sun, are so chosen that the areas of the *sectors* swept out by the body are proportional to the time intervals, then the dynamical condition is also fulfilled. The double areas of the triangles are (in the usual notation)

$$(2) \quad [r_1r_2] = -g_1\sqrt{p} \qquad [r_1r_3] = g_2\sqrt{p} \qquad [r_2r_3] = g_3\sqrt{p},$$

and the values of $-g_1$, g_2, and g_3, to which they are proportional, are given in terms of the time intervals, radius vector r_2, etc., by the well-known expressions †

* See L. J. Comrie, *Pop. Astron.*, **32**, 243. For references to suitable tables, see *M.N.*, **85**, 386.

† J. F. Encke, *Berl. Astron. Jahrbuch*, 1833, p. 288; J. C. Watson, *Theoretical Astronomy*, p. 176; E. Weiss, *Über die Bestimmung der Bahn eines Himmelskörper aus drei Beobachtung*, Denkschr. Akad. d. Wiss., Wien (Math. Classe), Bd. 60 (1893), p. 352; J. Bauschinger, *Bahnbestimmung der Himmelskörper* (Leipzig, 1906), p. 177.

$$(3)\quad \begin{cases} -g_1=\tau_3\left(1-\frac{\tau_3^2}{6r_2^3}-\frac{\tau_3^3}{4}\chi+\frac{\tau_3^4}{120r_2^6}+\frac{3\tau_3^4}{40}\Psi\ \ldots\right) \\ g_2=\tau_2\left(1-\frac{\tau_2^2}{6r_2^3}-\frac{\tau_2^2(\tau_3-\tau_1)}{4}\chi+\frac{\tau_2^4}{120r_2^6}+\frac{\tau_2^2(3\tau_2^2-10\tau_1\tau_3)}{40}\Psi\ \ldots\right) \\ g_3=\tau_1\left(1-\frac{\tau_1^2}{6r_2^3}+\frac{\tau_1^3}{4}\chi+\frac{\tau_1^4}{120r_2^6}+\frac{3\tau_1^4}{40}\Psi\ \ldots\right), \\ \text{where}\quad \chi=\frac{1}{r_2^4}\frac{dr_2}{d\tau}\qquad \Psi=\frac{1}{r_2^4}\left[\left(\frac{d^2r_2}{d\tau^2}\right)-\frac{4}{r_2}\left(\frac{dr_2}{d\tau}\right)^2\right], \\ \text{then}\quad c_1=\frac{[r_2r_3]}{[r_1r_3]}=\frac{g_3}{g_2}\qquad\text{and}\qquad c_3=\frac{[r_1r_2]}{[r_1r_3]}=\frac{-g_1}{g_2}. \end{cases}$$

By carrying out the division we get

$$(4)\quad \begin{cases} c_1=\frac{\tau_1}{\tau_2}\left\{1+\frac{\tau_2^2-\tau_1^2}{6r_2^3}+\frac{\tau_3(\tau_2\tau_3-\tau_1^2)}{4}\chi\right. \\ \qquad\left.+\frac{7\tau_2^4-10\tau_1^2\tau_2^2+3\tau_1^4}{360r_2^6}-\frac{3(\tau_2^4-\tau_1^4)-10\tau_1\tau_2^2\tau_3}{40}\Psi\ \ldots\right\} \\ c_3=\frac{\tau_3}{\tau_2}\left\{1+\frac{\tau_2^2-\tau_3^2}{6r_2^3}-\frac{\tau_1(\tau_1\tau_2-\tau_3^2)}{4}\chi\right. \\ \qquad\left.+\frac{7\tau_2^4-10\tau_2^2\tau_3^2+3\tau_3^4}{360r_2^6}-\frac{3(\tau_2^4-\tau_3^4)-10\tau_1\tau_2^2\tau_3}{40}\Psi\ \ldots\right\} \end{cases}$$

Neglecting the last two terms, these are arranged for practical working as follows:

$$(5)\quad \begin{cases} c_1{}^0=\frac{\tau_1}{\tau_2}\ \text{where the time intervals are:}\ \tau_1=k(t_3-t_2)\qquad k=0{\cdot}0172021 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tau_3=k(t_2-t_1)\quad \log k=8{\cdot}235581 \\ c_3{}^0=\frac{\tau_3}{\tau_2}\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tau_2=k(t_3-t_1) \\ \nu_1=\frac{\tau_2^2}{6}c_1{}^0(1-(c_1{}^0)^2)+\frac{\tau_2^3}{4}c_1{}^0(1-c_1{}^0)\{1-c_1{}^0-(c_1{}^0)^2\}\frac{1}{r}\frac{dr}{d\tau} \\ \qquad=\tau_2{}^2a_1+\tau_2{}^3b_1\frac{1}{r_2}\frac{dr}{d\tau},\quad\text{and similarly} \\ \nu_3=\tau_2{}^2a_3-\tau_2{}^3b_3\frac{1}{r_2}\frac{dr}{d\tau}. \end{cases}$$

The values of a_1, b_1, a_3, and b_3 may be taken from Table II. on p. 729, remembering that $c_1{}^0+c_3{}^0=1$. The $\frac{1}{r_2}\frac{dr}{d\tau}$ terms are usually very small in a preliminary orbit, and must in any case be neglected in the first approximation. Alternative equations for ν_1 and ν_3 are then

$$(5a)\quad \begin{cases} \nu_1=\frac{\tau_1\tau_3}{6}(1+c_1{}^0),\quad\text{and finally}\ c_1=c_1{}^0+\frac{\nu_1}{r_2{}^3} \\ \nu_3=\frac{\tau_1\tau_3}{6}(1+c_3{}^0)\qquad\qquad c_3=c_3{}^0+\frac{\nu_3}{r_2{}^3}. \end{cases}$$

Various other similar equations for these ratios have been published, but as they involve more than one radius vector they lead to greater complexity in the working, without adequate practical advantages. The very exact formulæ of Gibbs, which require a knowledge of all three radii vectores r_1, r_2, r_3, will be referred to later.

It is convenient before returning to (1) to mention one or two other equations required. We have for the heliocentric co-ordinates x, y, z

$$(6) \quad . \quad \begin{cases} x_1=\rho_1 l_1 - X_1 & \text{where} \quad l_1 = \cos\delta_1 \cos\alpha_1 \\ y_1=\rho_1 m_1 - Y_1 & \qquad\quad m_1 = \cos\delta_1 \sin\alpha_1 \\ z_1=\rho_1 n_1 - Z_1 & \qquad\quad n_1 = \sin\delta_1, \end{cases}$$

and similar formulæ for the second and third places. In these formulæ ρ_1, ρ_2, ρ_3 are the geocentric distances of the object (minor planet or comet) which it is required to find, and α_1, δ_1, α_2, δ_2, α_3, and δ_3 are the given Right Ascensions and Declinations corresponding to the observation times t_1, t_2, t_3. The solar co-ordinates X, Y, Z (*i.e.* earth's co-ordinates reversed) are interpolated from the usual astronomical ephemerides, and, as in the case of α and δ, should be referred to the equator of the mean equinox of the beginning of the year.

It is a very easy matter to eliminate the effects of parallax completely by applying the geocentric rectangular co-ordinates of the observatories, with reversed signs, to the solar co-ordinates of each date respectively. Namely, add (in units of the seventh decimal place)

$$(7) \quad \begin{cases} \Delta X = A\cos\theta & \text{where} \quad \theta = \text{Local Sidereal Time.} \\ \Delta Y = A\sin\theta & \qquad A = -15{\cdot}10^7 \sin 1''(\pi_0\rho_E \cos\phi')^s \\ \Delta Z & \qquad \Delta Z = -10^7 \sin 1''(\pi_0\rho_E \sin\phi')'' \\ \log 15{\cdot}10^7 \sin 1'' = 2{\cdot}8617, & \qquad \log 10^7 \sin 1'' = 1{\cdot}6856. \end{cases}$$

The logarithms of the quantities $\frac{1}{1.3}(\pi_0\rho_E \cos\phi')$ and $(\pi_0\rho_E \sin\phi')$ for the various observatories will be found in the *Connaissance des Temps* for any recent year.* The values of log A and ΔZ for some of the observatories are given in Table I. at the end (p. 728).

We shall later require also

$$(8) \quad . \quad \begin{cases} R_1\cos\psi_1 = l_1X_1 + m_1Y_1 + n_1Z_1 & R_1^2 = X_1^2 + Y_1^2 + Z_1^2 \\ R_2\cos\psi_2 = l_2X_2 + m_2Y_2 + n_2Z_2 & R_2^2 = X_2^2 + Y_2^2 + Z_2^2 \\ R_3\cos\psi_3 = l_3X_3 + m_3Y_3 + n_3Z_3 & R_3^2 = X_3^2 + Y_3^2 + Z_3^2. \end{cases}$$

ψ, in each case, is the internal angle sun-earth-object.

Returning now to equation (1) and substituting (6) in it, we get immediately

$$(9) \quad . \quad . \quad \begin{cases} c_1\rho_1 l_1 - \rho_2 l_2 + c_3\rho_3 l_3 - X_1c_1 + X_2 - X_3c_3 = 0 \\ c_1\rho_1 m_1 - \rho_2 m_2 + c_3\rho_3 m_3 - Y_1c_1 + Y_2 - Y_3c_3 = 0 \\ c_1\rho_1 n_1 - \rho_2 n_2 + c_3\rho_3 n_3 - Z_1c_1 + Z_2 - Z_3c_3 = 0. \end{cases}$$

* Or J. Bauschinger, Table 32.

This is essentially the fundamental equation of the Gaussian method expressed in rectangular co-ordinates. It is necessary firstly to solve it for ρ_2.

At this stage it is possible, of course, to split c_1 and c_3 into their c^0 and ν components, and also to put the solution into determinant form, but such steps involve extra labour without apparently giving any appreciable advantages as compensation. There is much to be said for the straightforward process of writing down the equations (9) directly, treating $c_1\rho_1$, ρ_2, $c_3\rho_3$, c_1, and c_3 as unknown quantities and putting in the known numerical values of all the other quantities. After eliminating $c_1\rho_1$ and $c_3\rho_3$ in the usual way we obtain an equation of the form

$$\text{(10)} \quad . \quad . \quad . \quad . \quad \mathrm{E}\rho_2=\mathrm{F}_1c_1+\mathrm{F}_2+\mathrm{F}_3c_3,$$

in which E, F_1, F_2, and F_3 are known.

Then using c_1^0, c_3^0, ν_1, and ν_3 from (5), we obtain A and B from

$$\text{(11)} \quad . \quad . \quad . \quad \mathrm{EA}=c_1^0\mathrm{F}_1+\mathrm{F}_2+c_3^0\mathrm{F}_3,$$

$$\text{(12)} \quad . \quad . \quad . \quad . \quad \mathrm{EB}=\nu_1\mathrm{F}_1+\nu_3\mathrm{F}_3,$$

and hence get

$$\text{(13)} \quad . \quad . \quad . \quad . \quad \rho_2=\mathrm{A}+\frac{\mathrm{B}}{r_2{}^3},$$

which is the required dynamical relationship between ρ_2 and r_2. For the geometrical relationship between them we have, using (8)

$$\text{(14)} \quad . \quad . \quad \begin{cases} r_2{}^2=\rho_2{}^2-2\mathrm{R}_2\cos\psi_2\,.\,\rho_2+\mathrm{R}_2{}^2 \\ \quad =(\rho_2-\mathrm{R}_2\cos\psi_2)^2+\mathrm{R}_2{}^2\sin^2\psi_2. \end{cases}$$

The solution of these two equations, (13) and (14), may be made directly, a near value of ρ_2 being quickly found with the aid of a slide-rule. The possibility of a double solution must not be overlooked,* the two being found by plotting roughly the curve of $\rho_2-\mathrm{A}-\frac{\mathrm{B}}{r_2{}^3}$ $(=0)$ for different values of ρ_2. It must also be remembered that $\rho_2=0$, nearly, is one solution, so that $\mathrm{A}=-\frac{\mathrm{B}}{\mathrm{R}_2{}^3}$, nearly, offers an approximate check. This would be exact if the observations were made from the centre of gravity (barycentre) of the earth-moon system, which is the point describing an ellipse round the sun, whereas actually our data are referred to the positions in space of the observatories.

The equations (13) and (14) may be transformed into the usual Gaussian form as follows:

$$\text{(15)} \quad . \quad . \quad \begin{cases} \text{Put} \quad a\sin q=\mathrm{R}_2\sin\psi_2 \quad (\text{all}+ve) \\ \qquad a\cos q=\mathrm{A}-\mathrm{R}_2\cos\psi_2 \\ \qquad\qquad f=\dfrac{-\mathrm{B}}{a\mathrm{R}_2{}^3\sin^3\psi_2}, \end{cases}$$

* See C. V. L. Charlier, *M.N.*, **71**, 120 and 454; but regarding parabolic orbits he requires correction, see my remarks, p. 705, and reference p. 705 (†).

and we then solve *

$$(16) \quad . \quad . \quad . \quad . \quad \sin(z-q)=f\sin^4 z.$$

When the angle z has been found, we have

$$(17) \quad . \quad . \quad \rho_2=\frac{R_2}{\sin z}\sin(\psi_2+z) \qquad r_2=\frac{R_2\sin\psi_2}{\sin z}.$$

It would appear much quicker, however, to solve equations (13) and (14) directly. When a satisfactory value of ρ_2 has been obtained we then also have r_2 from (14), provisional values for c_1 and c_3 become known from (5*a*), and provisional values for ρ_1 and ρ_3 from (9), etc.

It will be found that in a preliminary orbit the quantities E, EA, and EB in (10), etc., are known only to a few significant figures, and that in consequence ρ_2 will not be obtainable from (13) to anything like the five- or six-figure accuracy to which it is desired to compute the elements in order to represent the given places. This arises from the limited length of the observed arc, owing to which the solution is always more or less indeterminate, and the difficulty being inherent in the problem, no method of using the given observations can be made to overcome it. Nevertheless, in order that the observations may be represented, it is necessary to make the adopted value of ρ_2 up to the accuracy required, say six figures, and with this to deduce ρ_1 and ρ_3, and subsequently the elements, to the same order of accuracy. There are infinitely many sets of elements *within a limited range*, and each set, if consistent in itself, will represent the given observations. Now though, in the case of a preliminary orbit, this range may only correspond to a knowledge of the elements to, say, a two- or three-figure accuracy, yet, in order to insure that the individual elements do not fall outside the range within which it is possible to determine them, it is necessary to obtain a representation of the observed places within the accuracy of the observations. This matter has been fully dealt with by F. R. Moulton in his memoir (*A.J.*, **28**, especially p. 107).

When the observed positions lie on a great circle, then in our equation (10) E=0, so that after inserting in it the values of c_1^0, c_3^0, ν_1, and ν_3 we use, instead of (13), the resulting equation of the form

$$(18) \quad . \quad . \quad . \quad . \quad . \quad A'+\frac{B'}{r_2{}^3}=0.$$

Such a value of ρ_2 has to be chosen (other than zero) as will lead, after obtaining ρ_1 and ρ_3 from (9), etc., to such values of r_1, r_2, r_3, that is r_2 (=1 nearly †) and $\frac{dr}{d\tau}$, as will satisfy equation (18) with revised values of ν_1 and ν_3 (*vide seq.*). Thus in general the problem, when the positions lie on a great circle, is *not* indeterminate by Gauss's method

* See also Th. Banachiewicz, *Tables Fondamentales pour la résolution de l'équation de Gauss . . .* (Gauthier-Villars, Paris, 1916).

† This is deducible from Lambert's theorem that the curvature of the apparent path is to or from the sun according as r is less than or greater than unity.

as here given,* but when in addition they lie on the ecliptic then A′ and B′ will also approach zero and a solution becomes *impossible by any method*. Under such circumstances four observations are required, and from these, two sets of three equations of the type (9) can be formed, in which two geocentric distances are common to both sets. Then, on eliminating the remaining geocentric distance in each set, we shall have four equations with two unknown distances, which can be combined so as to determine one distance, say ρ_2.

Correction for Planetary Aberration (*Light Time*).—As soon as preliminary values of ρ_1, ρ_2, and ρ_3 are available, each of the corresponding observation dates can be corrected for the light time:

(19) . t (corrected) $=t-\cdot 005770\rho$, in days $\log \cdot 005770 = 7\cdot 7612$.

As a rule, the corrections will not appreciably affect the ratios c_1^0, c_3^0, and if the value of the interval τ_2 is altered sufficiently to necessitate a revision of the values of ν_1 and ν_3 then this may be done by multiplying each of them by the factor $\dfrac{\tau_2^2\,(\text{corrected})}{\tau_2^2\,(\text{uncorrected})}$. We note that, with sufficient accuracy,

$$(20)\quad \frac{\tau_2^2\,(\text{corrected})}{\tau_2^2\,(\text{uncorrected})} = 1 - \cdot 01154\frac{\rho_3-\rho_1}{t_3-t_1} \quad \text{or,}$$

$$\log \frac{\tau_2^2\,(\text{corrected})}{\tau_2^2\,(\text{uncorrected})} = -\cdot 00501\frac{\rho_3-\rho_1}{t_3-t_1} \qquad \log \cdot 00501 = 7\cdot 700.$$

A similar correction to B in (13), which depends on ν_1 and ν_3, is rarely warranted owing to the uncertainty, already referred to, in the values of the coefficients A and B. *In most cases, therefore, the first solution of* ρ_2 *is final*, although it may be necessary to revise the values of ρ_1 and ρ_3 deduced from the *adopted* (six-figure) value of ρ_2 on account of corrections made to c_1 and c_3.

The Correction of c_1 *and* c_3.—In the case of small perihelion distances, and where the interval exceeds, say, a week, it may no longer be possible to neglect the $\dfrac{dr}{d\tau}$ terms in (5). It then becomes necessary to have recourse to more accurate formulæ for the ratios c_1 and c_3, such as Gibbs's, or to introduce the value of $\dfrac{dr}{d\tau}$ together with the corrected τ_2^2, just mentioned, into our formulæ (5). This we may do as follows. From our provisional values of ρ_1 and ρ_3 and with (8) we compute

$$(21)\quad r_1^2 = \rho_1^2 - 2R_1\cos\psi_1\,.\,\rho_1 + R_1^2 \quad \text{and} \quad r_3^2 = \rho_3^2 - 2R_3\cos\psi_3\,.\,\rho_3 + R_3^2,$$

and then from r_1, r_2, r_3 interpolate a value for $\dfrac{dr}{d\tau}$ for the middle date—for example

$$(22)\quad \frac{dr}{d\tau} = \frac{r_3-r_1}{\tau_2},$$

and introduce it into (5).

* Leuschner's dictum, *Lick Obs. Pubns.*, vol. vii., p. 224, requires modification.

It will be seen from (4) that the errors due to the neglected terms of the next higher order are of the fourth order of time intervals. In the well-known formulæ of Gibbs, which follow,* they are practically of the fifth order, or of the sixth if the time intervals are equal. Let

$$(23)\quad \mu_1=\tfrac{1}{12}(\tau_2\tau_3-\tau_1^2) \qquad \mu_2=\tfrac{1}{12}(\tau_1\tau_3+\tau_2^2) \qquad \mu_3=\tfrac{1}{12}(\tau_1\tau_2-\tau_3^2).$$

Table III. gives $\mu_1 : \tau_2^2$, $\mu_2 : \tau_2^2$, and $\mu_3 : \tau_2^2$ with argument c_1^0. Then

$$(24)\quad \begin{cases} c_1=c_1^0\dfrac{1+\mu_1 r_1^{-3}}{1-\mu_2 r_2^{-3}} & c_3=c_3^0\dfrac{1+\mu_3 r_3^{-3}}{1-\mu_2 r_2^{-3}} \\ \text{corrected } \nu_1=(c_1-c_1^0)r_2^3 & \nu_3=(c_3-c_3^0)r_2^3. \end{cases}$$

For the neglected higher order terms Weiss (*loc. cit.* p. 355) gives as the correction to be added to c_1 and subtracted from c_3

$$(25)\quad +\frac{\tau_1\tau_3(\tau_1+\tau_2)(\tau_2+\tau_3)(\tau_3-\tau_1)}{40\tau_2 r_2^4}\left[\frac{1}{9r_2^2}-\frac{d^2r_2}{d\tau^2}+\frac{4}{r_2}\left(\frac{dr_2}{d\tau}\right)^2\right].$$

Writing this

$$(26)\quad w\,\frac{\tau_2^4}{r_2^6}\left[\tfrac{1}{9}-r_2^2\frac{d^2r}{d\tau^2}+4r_2\left(\frac{dr}{d\tau}\right)^2\right],$$

the values for w run as follows :

$c_1^0=0{\cdot}10$	$w=0{\cdot}0038$	$c_1^0=0{\cdot}50$	$w=\ 0{\cdot}0000$
·20	·0052	·60	$-$ ·0027
·30	·0046	·70	$-$ ·0046
·40	·0027	·80	$-$ ·0052
·50	·0000	·90	$-$ ·0038

Similarly for c_3^0, but with signs reversed. The maximum value is for $c_1^0=0{\cdot}22$ or $0{\cdot}78$, for which $w=\pm{\cdot}0052$, say $\frac{1}{200}$.

The expression (26) can be put into another form. From the well-known equations

$$(27)\quad p=r^2V^2-\left(r\frac{dr}{d\tau}\right)^2$$

$$(28)\quad V^2=\frac{2}{r}-\frac{1}{a}$$

we get

$$(29)\quad \left(\frac{dr}{d\tau}\right)^2=\frac{2r-p}{r^2}-\frac{1}{a},$$

and by differentiating

$$(30)\quad \frac{d^2r}{d\tau^2}=\frac{p-r}{r^3},$$

so that (26) becomes

$$(31)\quad w\,\frac{\tau_2^4}{r_2^6}\left[\frac{82}{9}-\frac{5p}{r}-\frac{4r}{a}\right].$$

* Gibbs, "On the Determination of Elliptical Orbits," *Nat. Acad. of Science*, iv. (8 mem.), 1888. See also E. Weiss, *loc. cit.*, p. 355 ; J. Bauschinger, *loc. cit.*, p. 179.

For the *circle* $p=a=r$, and the *maximum* value of (31) becomes

$$(32) \quad \ldots \quad \pm\frac{1}{1800}\cdot\frac{{\tau_2}^4}{{r_2}^6}.$$

For the *parabola* $p=2q$, $a\to\infty$ and the *maximum* value of (31) becomes

$$(33) \quad \ldots \quad \pm\frac{1}{20}\cdot\frac{{\tau_2}^4}{{r_2}^6}\left(\frac{82}{90}-\frac{q}{r}\right).$$

A similar investigation of the higher order terms neglected in (5) indicated corrections which do *not* vanish for equal intervals and whose range is small for the usual time-interval ratios met with. An *average* value of the correction to be added both to c_1 and c_3 was found to be

$$(34) \quad \ldots \quad +\frac{1}{200}\frac{{\tau_2}^4}{{r_2}^6}\left(10-\frac{5p}{r}-\frac{4r}{a}\right),$$

or for the *circle*,

$$(35) \quad \ldots \quad +\frac{1}{200}\frac{{\tau_2}^4}{{r_2}^6},$$

or for the *parabola*,

$$(36) \quad \ldots \quad +\frac{1}{20}\frac{{\tau_2}^4}{{r_2}^6}\left(1-\frac{q}{r}\right).$$

The superiority of Gibbs's formulæ (24), especially for equal intervals, is clear. They are usually sufficient, in the case of minor planets, for arcs of one month or more, but for comets much will depend on the heliocentric distances involved. In this latter case, however, the special conditions of a parabola furnish us with simple and accurate formulæ for use with which a table is available.*

Let $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ represent the ratios of the sectors : triangles corresponding to the time intervals τ_1, τ_2, τ_3 respectively, then with

$$(37) \quad \ldots \quad \eta_1=\frac{2\tau_1}{(r_2+r_3)^{\frac{3}{2}}} \qquad \eta_2=\frac{2\tau_2}{(r_1+r_3)^{\frac{3}{2}}} \qquad \eta_3=\frac{2\tau_3}{(r_1+r_2)^{\frac{3}{2}}}$$

we obtain, preferably from the table mentioned (but see p. 724), the parabolic values of $\log \bar{y}$ corresponding to each. Then

$$(38) \quad \ldots \quad c_1={c_1}^0\frac{\bar{y}_2}{\bar{y}_1} \qquad c_3={c_3}^0\frac{\bar{y}_2}{\bar{y}_3}.$$

To return to the consideration of the general orbit. Even when the corrections to c_1 and c_3 are not sufficient to modify the equation (13) within the limits of accuracy to which the coefficients, A and B, are known, it will nevertheless be necessary, as has already been remarked, to recompute ρ_1 and ρ_3, starting from the adopted value of ρ_2, made up to the required (five- or six-figure) accuracy, and using the revised

* J. F. Encke, *loc. cit.*, p. 311 (his ν=my $\bar{y}$); J. Bauschinger, *Tafeln zur Theoretischen Astronomie* (Leipzig, 1901), Table XXIIa; C. Wirtz, *Tafeln und Formeln aus Astronomie und Geodäsie* (Berlin, 1918), Table 56.

values of c_1 and c_3 in order to obtain a correct solution consistent with this value of ρ_2.

With these final values of ρ_1, ρ_2, ρ_3, and the corrected times, it only remains to compute the heliocentric co-ordinates by (6), and obtain the elements in the usual way. Before doing so, however, it is wise as a check to recompute the middle place, using the heliocentric co-ordinates of the first and third places and the corrected values of c_1 and c_3 :

$$(39) \quad \begin{cases} \rho_2 \cos\delta_2 \cos\alpha_2 = x_2 + \mathrm{X}_2 = c_1 x_1 + c_3 x_3 + \mathrm{X}_2 \\ \rho_2 \cos\delta_2 \sin\alpha_2 = y_2 + \mathrm{Y}_2 = c_1 y_1 + c_3 y_3 + \mathrm{Y}_2 \\ \rho_2 \sin\delta_2 \qquad\quad = z_2 + \mathrm{Z}_2 = c_1 z_1 + c_3 z_3 + \mathrm{Z}_2. \end{cases}$$

The representation of the observation should be exact.

The Computation of the Elements.—Formulæ for obtaining the elements will be found in the summary at the end of this paper, but some mention should here be made of a convenient modification of the usual Gaussian equations used for computing ephemerides. We have

$$\begin{aligned} x &= r \sin a \sin(\mathrm{A}'+v), \\ &= \sin a \sin(\mathrm{A}+\omega) r\cos v + \sin a \cos(\mathrm{A}+\omega) \,.\, r \sin v \end{aligned}$$

or

$$(40) \quad . \quad . \quad . \quad x = \mathrm{P}_x \,.\, r\cos v + \mathrm{Q}_x \,.\, r \sin v.$$

Here, as in the formulæ which follow, similar equations for y and z, involving the constants P_y, Q_y, P_z, and Q_z respectively, are to be understood. We then have for the *general* orbit,*

$$(41) \quad . \quad x = a\mathrm{P}_x(\cos\mathrm{E} - e) + b\mathrm{Q}_x \sin\mathrm{E} \qquad b = a\cos\phi = a\sqrt{1-e^2},$$

and for the *parabola*,†

$$(42) \quad . \quad . \quad . \quad x = q\mathrm{P}_x\left(1 - \tan^2\frac{v}{2}\right) + 2q\mathrm{Q}_x \tan\frac{v}{2}.$$

It is very convenient to obtain the six constants P_x, P_y, P_z, and Q_x, Q_y, and Q_z directly from the heliocentric co-ordinates, when the true anomalies v_1 and v_3 are known, without first computing the elements ω, Ω, i. We have

$$(43) \quad . \quad . \quad . \quad \begin{cases} x_1 = \mathrm{P}_x r_1 \cos v_1 + \mathrm{Q}_x r_1 \sin v_1 \\ x_3 = \mathrm{P}_x r_3 \cos v_3 + \mathrm{Q}_x r_3 \sin v_3, \end{cases}$$

containing only two unknowns, P_x and Q_x. Consequently we can write out simple formulæ to give the constants we require, but where short arcs are in question it will be found that the six constants obtained in this way will not be consistent to the required accuracy with a single set of the three elements ω, Ω, and i. This arises from the fact that their determination rests on the differences between positions near together, *i.e.* the small differences between two nearly equal quantities, and in order to overcome this, extra significant figures have to be

* See *B.A.A. Journal*, **32**, C. E. Adams, p. 231, and L. J. Comrie, p. 234.
† A. C. D. Crommelin, *B.A.A. Journal*, **32**, 305.

inserted, and where v is small this may necessitate the use of seven- or eight-figure tables. An easy transformation overcomes these difficulties and leads to the following equations here adopted: *

$$r_1 r_3 \cos(v_3 - v_1) = x_1 x_3 + y_1 y_3 + z_1 z_3. \tag{44}$$

$$\sigma = \frac{r_1 r_3 \cos(v_3 - v_1)}{r_1^2}. \tag{45}$$

$$\bar{x} = x_3 - \sigma x_1, \tag{46}$$

etc., made up to the (six-figure) accuracy required.

$$\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2. \tag{47}$$

$$Q = r_1 r_3 \sin(v_3 - v_1) = r_1 \bar{r}. \tag{48}$$

We note that, as a check,

$$Q^2 = (y_1 z_3 - y_3 z_1)^2 + (x_1 z_3 - x_3 z_1)^2 + (x_1 y_3 - x_3 y_1)^2, \tag{49}$$

but if the two values of Q differ slightly, that from (48) should be used to obtain v_1 and v_3. Then

$$P_x = x_1 \frac{\cos v_1}{r_1} - \bar{x} \frac{\sin v_1}{\bar{r}}, \quad \text{etc.} \tag{50}$$

$$Q_x = x_1 \frac{\sin v_1}{r_1} + \bar{x} \frac{\cos v_1}{\bar{r}}, \quad \text{etc.} \tag{51}$$

Before leaving the general orbit, mention may be made of a simple formula given by Bauschinger for obtaining an approximate value of the semi-major axis a from three radii vectores and the time intervals †

$$\frac{\tau_1 \tau_3}{a} = r_2^2 \left(1 + \frac{2\mu_2}{r_2^3}\right) - c_1^0 r_1^2 \left(1 - \frac{2\mu_1}{r_1^3}\right) - c_3^0 r_3^2 \left(1 - \frac{2\mu_3}{r_3^3}\right), \tag{52}$$

where the values of μ_1, μ_2, and μ_3 are the same as in (23).

The Parabolic Orbit.—For the determination of this special type of orbit we introduce an extra condition with the fundamental equations (9), namely, that the eccentricity shall be unity. It does not appear practicable, however, to do this directly, and the usual method of procedure is to obtain from the three given equations, or their equivalents, a relationship between two of the geocentric distances, and using this, to find such a value of the chord as will also satisfy Lambert's equation for the parabola (*i.e.* Euler's equation) which gives the relationship between the chord, the two radii vectores to the sun, and the time interval. The equation is

$$(r_1 + r_3 + s)^{\frac{3}{2}} - (r_1 + r_3 - s)^{\frac{3}{2}} = 6\tau_2, \tag{53}$$

in which s is the chord which is given geometrically by

$$s^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2. \tag{54}$$

* Practically the same as given by C. Veithen, *loc. cit.*, p. 16. For another form see: Th. Banachiewicz, *Circ. de l'Observ. de Cracovie*, **20**, 3.

† J. Bauschinger, *Bahnbestimmung*, p. 185.

The equation (53) is usually used in Encke's modified form:

$$(55) \quad . \quad . \quad \eta=\frac{2\tau_2}{(r_1+r_3)^{\frac{3}{2}}} \quad \text{then} \quad s=\eta\mu(r_1+r_3),$$

where μ is taken from a table for this purpose, with argument η.* The solution is completed when the values of the chord from (54) and (55) agree exactly.

The introduction of the extra condition means that the number of elements of the orbit is reduced from six to five, and that in consequence we do not use the whole of the three observations fully. Any two of the three equations of (9) will give us the required relationship between two of the geocentric distances, but it is best to use those showing the greatest motion in the direction cosines, l, m, or n, in order that the chief components of the apparent motion shall be represented in the solution. The equation resulting from the elimination of one geocentric distance from the two chosen equations of (9) will require a knowledge of $1 : r_2^3$ which is involved in c_1 and c_3. In general it is not possible to neglect the $1 : r_2^3$ terms if a satisfactory representation of the places is desired,† and the first hypothesis in Olbers's method, which does so, as a rule leaves something to be desired.

Two methods of obtaining the required result suggest themselves: (*a*) to use the first and third distances and obtain r_2 by interpolation from r_1 and r_3 ($r_2=c_1^0 r_1+c_3^0 r_3$ will usually suffice for very short arcs); or (*b*) to use the second and third distances (or the first and second), in which case there is the advantage that r_2 is obtained directly from each trial value of ρ_2, but the slight disadvantage, however, that in order to obtain the best result a small adjustment must finally be made, so that the first and third places are exactly satisfied, and the residuals, if any, thrown into the middle place. In short arcs, where the $1 : r_2^3$ term is anyhow small and where r_2 can easily be interpolated with sufficient precision, the former method is the more convenient. The derivation of most of the equations is quite straightforward, and they need not be detailed now as they are given in the summary of formulæ at the end. The following may, however, be noted here; using (3), etc., we put

$$(56) \quad . \quad . \quad . \quad \begin{cases} \dfrac{c_1}{c_3}=\dfrac{\tau_1}{\tau_3}\left(1-\dfrac{\tau_1^2-\tau_3^2}{6r_2^3}+\ \ldots\right) \\ \quad =\dfrac{c_1^0}{c_3^0}+\dfrac{\lambda_0}{r_2^3} \end{cases}$$

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* J. F. Encke, *Berl. Astron. Jahrbuch*, 1833, pp. 267 and 311; J. C. Watson, *Theor. Astron.*, Table XI.; J. Bauschinger, *Bahnbestimmung*, p. 194, *Tafeln*, xxii; C. Wirtz, *loc. cit.*, Table 55. See also other textbooks, many of which give this table. This μ is not to be confused with the same symbol used for the mean motion in an ellipse, or the μ_1, μ_2, μ_3 in (23).

† To use all three of the equations (9) and eliminate $1 : r_2^3$ as if it were an unknown, additional to ρ_2, would overcome this difficulty but would usually not lead to a satisfactory distribution of residuals.

and

$$(57)\quad \begin{cases} \dfrac{1}{c_3} = \dfrac{\tau_2}{\tau_3}\left(1 - \dfrac{{\tau_2}^2 - {\tau_3}^2}{6{r_2}^3} + \ldots\right) \\ \quad = \dfrac{1}{{c_3}^0} - \dfrac{\lambda_3}{{r_2}^3}, \end{cases}$$

and the values of λ_0 and λ_3, which we require in the equations for the first method, are

$$(58)\quad \begin{cases} \lambda_0 = \dfrac{{c_1}^0}{{c_3}^0} \cdot \dfrac{{c_3}^0 - {c_1}^0}{6} {\tau_2}^2 \\ \quad = \dfrac{{c_1}^0}{{c_3}^0}(\nu_1 - \nu_3) + \nu_1 - \nu_3 \left(\dfrac{{c_1}^0}{{c_3}^0}\right)^2. \end{cases}$$

$$(59)\quad \lambda_3 = \frac{\nu_3}{({c_3}^0)^2}.$$

$\lambda_0 : {\tau_2}^2$ and $\lambda_3 : {\tau_2}^2$ may be taken from Table IV. with argument ${c_1}^0$.

The interpolation of r_2, necessitated by this method, can be made more exact by including the next term and writing

$$(60)\quad r_2 = {c_1}^0 r_1 + {c_3}^0 r_3 - \tfrac{1}{2}\tau_1\tau_3 \frac{d^2r}{d\tau^2}.$$

From (30) we have for the parabola, where $p = 2q$ and $a \to \infty$,

$$(61)\quad \frac{d^2r}{d\tau^2} = \frac{2q - r}{r^3},$$

and from (27) and (28),

$$(62)\quad q = r - \tfrac{1}{2}r^2\left(\frac{dr}{d\tau}\right)^2.$$

This last equation may be used to determine an approximate value of the perihelion distance q when r_1 and r_3 are known, for if we identify r with the moment $\frac{1}{2}(t_3 + t_1)$ we can write (for short arcs)

$$(63)\quad \begin{cases} r = \dfrac{r_1 + r_3}{2} - \tfrac{1}{8}{\tau_2}^2 \dfrac{d^2r}{d\tau^2} \\ \quad = \dfrac{r_1 + r_3}{2} - \dfrac{{\tau_2}^2}{8r^3}(2q - r) \end{cases}$$

and

$$(64)\quad \frac{dr}{d\tau} = \frac{r_3 - r_1}{\tau_2}.$$

The first trial must necessarily neglect $\dfrac{d^2r}{d\tau^2}$, but a revision using an approximate value of q obtained from (62) will lead to a more accurate one.

For the purpose of interpolating r_2 we can substitute (62) in (61) and obtain

$$(65)\quad \frac{d^2r}{d\tau^2} = \frac{1}{r^2}\left\{1 - r\left(\frac{dr}{d\tau}\right)^2\right\}$$

and can then use for (60) the approximation

$$(66)\quad \left\{\begin{aligned} r_2 &= c_1^0 r_1 + c_3^0 r_3 - \frac{\tau_1\tau_3}{2r_2^2}\left(1 - r_2\frac{(r_3-r_1)^2}{\tau_2^2}\right) \\ &= c_1^0 r_1 + c_3^0 r_3 - \frac{c_1^0 c_3^0}{2r_2^2}\left(\tau_2^2 - r_2(r_3-r_1)^2\right). \end{aligned}\right.$$

One or two points of general importance may now be noted. If the given data truly represent a parabola then the parabolic solution we find will exactly satisfy all three observations; that is, it will satisfy equation (9) within the accuracy of the observational data. If this is not so then it indicates either that there is an error in the working or in the observations, or that there is a double solution, or that the observations are not compatible with a parabolic hypothesis for the orbit. In this last case we must revert to the solution of the general orbit, using the whole of the equations (9). The ease with which the parabolic solution can be tested by (39) without having to compute all the elements first, and with which the change to a general solution can be made, indicates one of the advantages of the methods here given.

The possibility that there may be a double solution with the two equations selected from (9) must not be overlooked, but in general there is only one solution that will completely satisfy the *whole* of the three observations. In exceptional cases, however, the double solution may be such that the given observations will not distinguish between them.*

Professor Th. Banachiewicz has shown,† moreover, that under certain special circumstances, where a general solution is altogether indeterminate, there may be *three* possible parabolic solutions. His fictitious example is worth quoting. Given

	(1)	(2)	(3)
t (days)	−10·000000	0·000000	+10·000000
X	+ 0·9852408	1·0000000	+ 0·9852408
Y	− 0·1711739	0·0000000	+ 0·1711739
λ	+0° 45′ 0″·0	0° 0′ 0″·0	−0° 45′ 0″·0

and β_1, β_2, β_3=0° 0′ 0″·0, then either of the following three sets of elements satisfy the given positions exactly, within the accuracy of a seven-figure computation:

	(1)	(2)	(3)
T	0	0	0
$\omega=\Omega$	0	0	0
i	0	0	180° 0′ 0″
log q	0·4231815	0·8497197	1·2646811

To return to the discussion of the two methods given above. The

* *E.g.* the preliminary orbit of Comet 1910 I. A double solution, one direct, one retrograde, in the case of Comet 1925*a* (compare the first orbits by Kobold and Crommelin) presumably arose from the indeterminacy of the general solution, as in the fictitious example by Banachiewicz, next mentioned.

† Th. Banachiewicz, "Sur un théorème de Legendre relatif à la détermination des orbites cométaires," *Bull. de l'Acad. Polonaise des Sciences*, Série A. Cracovie, 1924.

work involved in computing the trials in either of the ways is admittedly heavier than in the corresponding step in a *single* hypothesis in Olbers's method; but it must be remembered, apart from other points, that whereas in that method it is necessary to recompute most of the whole working two or even three times with new trial values of the ratios of the distances in order to obtain a satisfactory fit, in the method here given a final result is usually reached in the *first* attempt. Nevertheless it must be admitted that the work of solving the geocentric distances in a parabolic orbit by the methods here given is greater than in the case of the general orbit, and some means of shortening the process still further is very desirable.

Comparison with other Methods.—As will readily be seen, the method for the general orbit is essentially that of Gauss modified by using equations (5*a*) as recommended by Encke, as these are more accurate than those originally used by Gauss. There is a great advantage in using the equatorial co-ordinates, the transformation to the ecliptic ones being saved and the effects of parallax eliminated by a simple adjustment of the solar co-ordinates. Moreover, in the case of a cometary object, should the hypothesis as to the parabolic nature of the orbit prove incorrect, the general solution can then be carried out without recomputing any of the quantities used.

The methods at present in general use for the determination of preliminary orbits are those of Gauss (Encke's modification) * for planets or of Olbers for parabolic orbits, of Lagrange as arranged by Professor H. Andoyer,† and of Laplace as arranged by Professor A. O. Leuschner.‡ Judging by the expressions of opinion, published and unpublished, the greater number of computers are in favour of some form of Gauss's method for general orbits or Olbers's method for parabolic ones as being the most satisfactory in practice. Both Professor J. Bauschinger and the late Dr. F. Cohn, § well known for their experience in such work, have examined the matter critically and as impartially as possible, and have concluded in favour of the methods of Gauss and Olbers.

Professor Leuschner, however, has endeavoured to make the Laplacean method practicable, and he considers that the method he has developed is superior to the others, but his claims for its shortness and advantages seem on examination to be exaggerated. It is true that the first solution is quickly obtainable, but when tested it fails to represent the observations exactly, so that in every case a differential correction is necessary, and this extra work more than offsets the shortness of the preliminary solution. ‖ Too much stress is usually laid on

* J. Bauschinger, *Bahnbestimmung*, p. 278, and for "Olbers's method," p. 356.

† H. Andoyer, *Mécanique Céleste*, i. (Paris, 1923), chap. vii., etc.

‡ *Lick Obs. Pubns.*, vol. vii.

§ J. Bauschinger, "Über die Laplacesche Methode der Bahnbestimmung im Vergleich zur Gauss'schen," *Heinrich Weber Festschrift* (Teubner, Leipzig, 1912); Fritz Cohn, "Neue Methoden der Bahnbestimmung," *Vierteljahrsschrift der Astron. Gesellschaft* (Leipzig), 1918, pt. i. See also F. R. Moulton, *A.J.*, **28**, 110 and 118.

‖ My first orbit computations were made by Leuschner's method, and the disadvantages only became clear to me *later* when I tried the usual forms of Gauss's and Olbers's methods.

the fact that so much of the work can be done with fewer significant figures than in other methods. Of what value is this when innumerable small quantities have to be dealt with, whereby the chances of error are increased, and where the total labour is more and the checks fewer? The fact (*Lick Obs. Pubns.*, vii., 382 to 388) that slightly better results have frequently been obtained must not be allowed too much weight, since there is no doubt that the other methods mentioned, when properly handled, will represent the observations within the errors of observation, and no method can do more than this. Clearly the quality of the observations is not always the same, and doubtless there are many published orbits which do not represent the observations as well as they might, though at the time they were sufficient for the immediate practical purposes. It must, however, be admitted that in the case of a parabolic orbit Leuschner's method may lead to a better distribution of residuals than Olbers's.

It seems desirable, lest it might mislead anyone, to draw attention to an unfortunate statement made by Leuschner (*loc. cit.*, p. 24) in the example on Planet 1900 GA, "The geocentric distance is therefore practically indeterminate (by six-figure computation) by these methods (Gauss and Oppolzer). In a seven-place calculation it is nearly so, and the results of the first hypothesis do not furnish data sufficiently accurate for a second hypothesis, etc." Dr. F. Cohn (*loc. cit.*, p. 124) dealt with this example and revealed that this statement was quite incorrect. A seven-figure computation by Gauss's method was carried out by Dr. Stracke, after first eliminating parallax. The first hypothesis yielded O−C, 0″·0, +0″·6, a result practically as good as that obtained in the final solution by Leuschner's method after a lengthy computation, while a second hypothesis yielded a better fit even than his, though this was of no practical importance owing to the uncertainty in the observations.

This example was treated by the method in this paper, using the observations given (*loc. cit.*, p. 27), and the solar co-ordinates corrected by (7) to eliminate parallax. The fundamental equations (9) were then found to be

$$(67)\quad \begin{cases} +\cdot0129844\,c_1\rho_1 - \cdot0072509\,\rho_2 + \cdot0018910\,c_3\rho_3 + \cdot1246710\,c_1 \\ \qquad\qquad - \cdot1586625 + \cdot1917101\,c_3 = 0 \\ -\cdot9238459\,c_1\rho_1 + \cdot9212379\,\rho_2 - \cdot9186097\,c_3\rho_3 - \cdot9257635\,c_1 \\ \qquad\qquad + \cdot9214008 - \cdot9160994\,c_3 = 0 \\ -\cdot3825444\,c_1\rho_1 + \cdot3889320\,\rho_2 - \cdot3951615\,c_3\rho_3 - \cdot4015737\,c_1 \\ \qquad\qquad + \cdot3996801 - \cdot3973800 = 0, \end{cases}$$

and after eliminating $c_1\rho_1$ and $c_3\rho_3$ from these,

$$(68)\quad \cdot0001311\,\rho_2 = -\cdot1252509\,c_1 + \cdot1592388 - \cdot1922827\,c_3,$$

and putting in the values

$$(69)\quad \begin{cases} c_1{}^0 = \cdot4944064 & \nu_1 = \cdot0002964 \\ c_3{}^0 = \cdot5055936 & \nu_3 = \cdot0002986, \end{cases}$$

then

$$\cdot 0001311\rho_2 = +\cdot 0000971 - \cdot 0000945\frac{1}{r_2{}^3},$$

or

$$(70) \quad \begin{cases} \rho_2 = \cdot 7407 - \cdot 7208\frac{1}{r_2{}^3} & \text{and from (14)} \\ r_2{}^2 = \rho_2{}^2 + 2\cdot 010856\rho_2 + 1\cdot 033897. \end{cases}$$

The solution of these gave $\rho_2 = \cdot 5529$. This may be compared with the solutions by Leuschner's method (*loc. cit.*, pp. 29, etc.). The first gave $\rho_2 = \cdot 463$. Instead of proceeding straight to a differential correction, which, as was pointed out, would have been the correct course, two corrections for parallax were made, the second of which led to $\rho_2 = \cdot 5207$, and on comparison with the observations gave residuals of the order of 4″. A differential correction then gave the result finally adopted, $\rho_2 = \cdot 5503$, but left residuals of about 0″·3 in approximately the same direction as before, thus indicating that the definitive value *from the given data* must be very near $\rho_2 = \cdot 553$, which is exactly the first (and final) value here obtained!

A six-figure logarithmic calculation gave instead of (70)

$$(71) \quad \rho_2 = \cdot 735 - \cdot 7206\frac{1}{r_2{}^3}.$$

Thus even by a six-figure computation a solution is obtainable by the modified Gaussian method, and in fact the added accuracy of a seven-figure computation is purely illusory, whatever the method employed, owing to the observational uncertainties involved.

As regards the accuracy of the first solution for the geocentric distance Leuschner (*loc. cit.*, p. 222) quotes H. Poincaré (*Bull. Astr.*, **23**, 178, *et seq.*) as having shown that in the first hypothesis Laplace's method is superior to that of Gauss, inasmuch as for unequal intervals Laplace's error in the geocentric distance is only of the second order as compared with Gauss's error of the first order. This statement is misleading, since it is only true when the assumptions regarding the ratio of the triangles originally given by Gauss are employed, and Poincaré and Leuschner do not make it clear that computers now using Gauss's method adopt, *not* these formulæ, but the more accurate ones of Encke,* as given here in equation (5*a*).†

The method given by Professor H. Andoyer in his *Mécanique Céleste* (chap. vii., pp. 148, etc.) utilises fundamental equations similar to (9). In the notation of this paper they are:

$$(72) \quad \begin{cases} l_1 g_3 \rho_1 - l_2 g_2 \rho_2 - l_3 g_1 \rho_3 - g_3 X_1 + g_2 X_2 + g_1 X_3 = 0 \\ m_1 g_3 \rho_1 - m_2 g_2 \rho_2 - m_3 g_1 \rho_3 - g_3 Y_1 + g_2 Y_2 + g_1 Y_3 = 0 \\ n_1 g_3 \rho_1 - n_2 g_2 \rho_2 - n_3 g_1 \rho_3 - g_3 Z_1 + g_2 Z_2 + g_1 Z_3 = 0. \end{cases}$$

* *Berl. Astron. Jahrbuch*, 1833, p. 288, and 1854, p. 324.
† See Bauschinger, *Bahnbestimmung*, p. 183.

After carrying out the process of eliminating $g_3\rho_1$ and $g_1\rho_3$ he obtains the equation

(73) . . . . . $\rho_2 g_2 \Delta = D,$

and putting

(74) . . . . . $g_2 = \tau_2 - \frac{{\tau_2}^3}{6{r_2}^3}$

as a first approximation, with similar expressions for g_1 and g_3, he obtains an equation of the form

(75) . . . . . $\rho_2 = P + \frac{Q + q\rho_2}{{r_2}^3},$

and as usual

(76) . . . $r_2^2 = \rho_2^2 - 2R_2 \cos\psi_2 \cdot \rho_2 + R_2^2.$

The former equation, (73), clearly contains g_1, g_2, or g_3 in all the terms of D (see his equation (10), p. 148), and had these been retained intact, then on dividing (73) through by g_2 and putting in, instead, the values of the ratios (5*a*)

$$c_1 = \frac{g_3}{g_2} = {c_1}^0 + \frac{\nu_1}{{r_2}^3} \qquad c_3 = \frac{-g_1}{g_2} = {c_3}^0 + \frac{\nu_3}{{r_2}^3},$$

he would have obtained the equation (13) adopted in this paper,

$$\rho_2 = A + \frac{B}{{r_2}^3}.$$

The two equations are therefore fundamentally the same except that, in so far as Andoyer's equation (75) *contains* ρ_2 *on the right-hand side as well, it is less convenient than the one here adopted.* His subsequent procedure for improving the values of g_1, g_2, and g_3 is somewhat lengthy, and is based upon the use of closed expressions instead of series. Some such expressions become necessary, however, for longer arcs where the original series are inadequate. The corresponding step in Gauss's method, namely the recomputation of c and ν from the ratios sector : triangle, is indicated in the summary of formulæ, p. 720.

For the parabolic orbit Andoyer (*loc. cit.*, p. 165) uses the general solution and then modifies the geocentric distance ρ_2 till the heliocentric co-ordinates yield the parabolic velocity, that is, satisfy the equation

(77) . . . $\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 - \frac{2}{r} = 0.$

To utilise Lambert's equation for the parabola appears to me to be the more direct way of achieving the desired result.

Several other methods for the determination of orbits have been examined, but wherever some advantage has been sought and obtained, it seems there is invariably some disadvantage which more than offsets it.

There will now follow examples of an elliptical and of a parabolic orbit, in condensed form, and also the summary of formulæ and tables.

They do not claim to give a final arrangement, and doubtless computers will introduce improvements and modifications to suit their own convenience according as they use logarithms or a calculating machine for the work. My personal choice is in favour of a mixture of these means of calculation, and where operations are equally quickly performed by logarithms or a machine, the logarithmic computation seems preferable as involving less nervous strain.

Example (1).—*Planet* 1924 TD (Baade).

The working is intended to be followed with the summary of formulæ (p. 717). The given data are three photographic observations from the Bergedorf and Greenwich Observatories, namely :

	G.M.T. (old).	α 1924·0.	δ 1924·0.	
	h m s	h m s	° ′ ″	
1924 Oct. 23	7 25 32	21 5 11·60	+15 28 33·8	Bergedorf.
Oct. 28	11 27 58	21 30 27·23	12 4 32·5	Greenwich.
Nov. 3	7 31 55	21 58 32·11	8 24 29·4	Greenwich.

			α.	δ.	θ.
			° ′ ″	° ′ ″	h m
I.	(1)	Oct. 23·30940	316 17 54·0	15 28 33·8	22 14·2
	(2)	28·47775	322 36 48·5	12 4 32·5	1 56·1
	(3)	34·31383	329 38 1·7	8 24 29·4	22 23·0

	(1)	(2)	(3)	corrected.		(1)	(2)	(3)
ΔX	−228	−233	−243		X	−·860286	−·810811	−·747072
ΔY	+113	−129	+109		Y	−·457649	−·526063	−·598100
ΔZ	−341	−332	−332		Z	−·198547	−·228209	−·259468

II.	(1)	(2)	(3)		
l	+·696735	+·776976	+·853538	$\log \tau_1$	9·001702
m	−·665853	−·593753	−·500091	$\log \tau_2$	9·277149
n	+·266836	+·209204	+·146224	$\log \tau_3$	8·948933
				τ_2^2	·035834

c_1^0 ·530339 ν_1 ·002277
c_3^0 ·469661 ν_1 ·002186.

III.	(1)	(2)	(3)
$2R\cos\psi$	−·695288	−·730743	−·752981
R^2	+·988956	+·986236	+·983164.

IV.

$$+\cdot696735c_1\rho_1 \quad -\cdot776976\rho_2 \quad +\cdot853538c_3\rho_3 = -\cdot860286c_1 + \cdot810811 \quad -\cdot747072c_3$$

$$-\cdot665853c_1\rho_1 \quad +\cdot593753\rho_2 \quad -\cdot500091c_3\rho_3 = -\cdot457649c_1 + \cdot526063 \quad -\cdot598100c_3$$

$$+\cdot266836c_1\rho_1 \quad -\cdot209204\rho_2 \quad +\cdot146224c_3\rho_3 = -\cdot198547c_1 + \cdot228209 \quad -\cdot259468c_3$$

$|l_1| > |m_1|, |n_1|$, hence we multiply the top line by $\frac{m_1}{l_1} = -\cdot955676$ and

add to the second, and also by $\frac{n_1}{l_1}=+\cdot382981$ and subtract from the third. This eliminates $c_1\rho_1$ and gives

V.

$$\begin{array}{llll} -\cdot148784\rho_2 & +\cdot315615c_3\rho_3 & = & -1\cdot279804c_1 \quad +1\cdot300936 \quad -1\cdot312059c_3 \\ +\cdot088363\rho_2 & -\cdot180665c_3\rho_3 & = & +0\cdot130926c_1 \quad -0\cdot082316 \quad +0\cdot026646c_3. \end{array}$$

Similarly, multiplying the first of these by ·572422 and adding to the second eliminates $c_3\rho_3$ and gives

$$+\cdot003196\rho_2 = -\cdot601662c_1 +\cdot662368 -\cdot724405c_3$$

$$= +\cdot003058 -\cdot002954\frac{1}{r_2^3}.$$

We then have to solve

$$\rho_2=\cdot9568-\cdot9243\frac{1}{r_2^3}$$

$$r_2^2=\rho_2^2+\cdot730743\rho_2+\cdot986236.$$

With a slide-rule

$$\begin{array}{llll} \rho_2=1\cdot00 & r_2^2=2\cdot72 & \text{giving } \rho_2=\cdot957-\cdot205=\cdot752 \\ \rho_2=0\cdot50 & r_2^2=1\cdot60 & \rho_2=\cdot957-\cdot454=\cdot503. \end{array}$$

Proceeding then to four-figure work with $\rho_2=\cdot5020$ the result is $\rho_2=\cdot5024$, which is found to satisfy the equations and is therefore the provisional solution.

Then

$$\frac{1}{r_2^3}=0\cdot49145 \qquad c_1=\cdot531458 \qquad c_3=\cdot470735.$$

These values inserted in the first intermediate equation give $c_3\rho_3=\cdot246791$, and the three original equations then give $c_1\rho_1=\cdot260702$, ·260702, ·260707 respectively. Hence the first solution yields

$$\begin{array}{llllll} \rho_1 & 0\cdot490541 & \rho_2 & 0\cdot502400 & \rho_3 & 0\cdot524267 \\ r_1^2 & 1\cdot570654 & r_2^2 & 1\cdot605767 & r_3^2 & 1\cdot652783. \end{array}$$

VI. The corrections for light times are −·000007 to $\log\tau_1$, $\log\tau_2$, and $\log\tau_3$, but those to ν_1, ν_3 (−6 units of the eighth decimal place), and B are quite negligible. The corrections to c_1^0 and c_3^0, respectively, amount to only −2 and +2 in the sixth place.

The corrections to ν_1 and ν_3, computed from $\frac{dr}{d\tau}=+\cdot1709$, give corrected

$$\nu_1=+\cdot002288 \quad \nu_3=+\cdot002168 \quad \text{and} \quad c_1=\cdot531461 \quad c_3=\cdot470728.$$

For Gibbs's expressions we have $\mu_1=\cdot000563$, $\mu_2=\cdot003730$, $\mu_3=\cdot000925$ and obtain $c_1=\cdot531463$, $c_3=\cdot470730$, in good agreement with the values just mentioned. The resulting corrected values $\nu_1=\cdot002291$, $\nu_3=\cdot002171$ are then used for a revision of the solution. The original value $\rho_2=\cdot5024$ would still satisfy the fundamental equations of IV. within

two or three units in the sixth place, so that for a preliminary orbit a revision of this value is hardly necessary. Nevertheless, using $\rho_2 = \cdot 9568 - \cdot 9235 \frac{1}{r_2^3}$ we obtain the value $\rho_2 = \cdot 5045$. Then with $\frac{1}{r_2^3} = \cdot 48978$, $c_1 = \cdot 531459$, $c_3 = \cdot 470726$ and the resultant solution gives

		(1)	(2)	(3)
	ρ	0·492593	0·504500	0·526459
	r^2	1·574098	1·609416	1·656737.
VII.	x	+1·203493	1·202795	1·196425
	y	+0·129654	0·226515	0·334823
	z	+0·329989	0·333752	0·336449.

The co-ordinates x_2, y_2, z_2 obtained from $x_2 = c_1 x_1 + c_3 x_3$, etc., only differ from those given above by one unit in the last place. The solution therefore satisfies the observations, provided the values of c_1 and c_3 are final.

VIII.

$r_2 r_3 \cos(v_3 - v_2)$	1·627187	$r_1 r_3 \cos(v_3 - v_1)$	1·594325	$r_1 r_2 \cos(v_2 - v_1)$	1·587058
κ_1	1·80557	κ_2	1·79143	κ_3	1·78290
h_1	·0007260	h_2	·0026392	h_3	·000591
$\bar{y}_1$	1·0008059	$\bar{y}_2$	1·0029231	$\bar{y}_3$	1·000656

Hence $c_1 = \cdot 531459$ and $c_3 = \cdot 470727$ are the final values. The completion of the check in VII. with these values gives

$$x_2 \ +1\cdot 202797 \qquad y_2 \ +\cdot 226516 \qquad z_2 \ +\cdot 333752$$
$$a_2 \ 322^\circ\ 36'\ 49''\cdot 5 \qquad \delta_2 \ 12^\circ\ 4'\ 32''\cdot 2$$

and the (o−c) residuals $-1''\cdot 0$, $+0''\cdot 3$.

IX.

σ	1·012850	$\bar{r}^2$	$\cdot 0419261_4$	Q	·256897
$\bar{x}$	−·0225330	$\bar{y}$	+·2035030	$\bar{z}$	+·002220
$\sin(v_3 - v_1)$	·159080	$r_1 r_2 \sin(v_2 - v_1)$	·120927	$r_2 r_3 \sin(v_3 - v_2)$	·136531

$$v_3 - v_1 \ \ 9^\circ\ 9'\ 12''\cdot 6 \qquad v_2 - v_1 \ \ 4^\circ\ 21'\ 26''\cdot 1 \qquad v_3 - v_2 \ \ 4^\circ\ 47'\ 46''\cdot 4.$$

X. $p = 1\cdot 852548$ and the checks confirm this value.

$$\log q_1 = \log e \cos v_1 = 9\cdot 678125 \qquad \log q_3 = \log e \cos v_3 = 9\cdot 642733,$$

hence

$$v_1 \ \ 22^\circ\cdot 38700 \qquad v_3 \ \ 31^\circ\cdot 54050,$$

and the check, p from v_2 and r_2, is correct.

$$\log \sin \phi \ \ 9\cdot 712156 \qquad \phi \ \ 31^\circ\ 1'\ 30''\cdot 4 \qquad \log a \ \ \cdot 401867.$$

XI.

	E_1	$12^\circ\cdot 7699_2$	E_3	$18^\circ\cdot 1468_4$	e ·515414
	M_1	$6\cdot 2425_1$	M_3	$8\cdot 9493_2$	e° 29·5310
	μ	$885''\cdot 538 = 0^\circ\cdot 245982_4$.			

XII.

r_1 1·254631 $\sin v_1$ +·380861
$\bar{r}$ 0·2047587 $\cos v_1$ +·924632

yield

P_x	+·928857	P_y	−·282974	P_z	+·239065
Q_x	+·263584	Q_y	+·958321	Q_z	+·110198.

XIII. With a=2·522706 and b=2·161815, we then have

$$x=+2\cdot343233\ (\cos E-e)+0\cdot569820\sin E$$
$$y=-0\cdot713860\ (\cos E-e)+2\cdot071713\sin E$$
$$z=+0\cdot603091\ (\cos E-e)+0\cdot238228\sin E.$$

The check, $aP_xbQ_x+\ldots$, shows a residual of about nineteen units of the sixth place, due to the accumulation of small errors multiplied by ab. It represents, however, only an error of one or two units of the last place in the constants P_x, Q_x, etc. A recomputation of x_1, y_1, z_1, x_3, y_3, and z_3 from these equations shows errors of only one or two units in the sixth place.

The computation of the middle place with

$$M_2=7^\circ\cdot51380 \quad E_2=15^\circ\cdot31241$$

gives

x_2	+1·202792	y_2	+0·226517	z_2	+0·333752
α_2	322° 36′ 48″·6	δ_2	+12° 4′ 32″·7		

and the (o−c) residuals −o″·1, −o″·2 !

XIV.

$\sin i \sin\omega$	+·331926	$\sin\Omega$	−·599158	$\cos\epsilon$	+·9174134 } 1924·0.
$\sin i \cos\omega$	−·280253	$\cos\Omega$	−·800629	$\sin\epsilon$	+·3979354 }

Check +·539672 = −cos i sin Ω = ·539670.

Collecting these and the other elements together, we have

Epoch 1924 Oct. 28·5 G.M.T. (Old Style).

M	7°	31′	12″·0		
ω	130	10	30·7 } 1924·0	$\log a$	·401867
Ω	216	48	35·0 }	μ	885″·538.
i	25	44	53·1 }		
ϕ	31	1	30·4		

Although these elements satisfy the observations here used, they are nevertheless some way from the truth. The reason is that the solution of the equations in V. is relatively indeterminate, the result being really only known to two- or three-figure accuracy.

Example (2).—*Comet* 1925 c (*Orkisz*).

The data were deduced from about ten observations which had been corrected for parallax (but *not* light time) on the basis of distances

taken from a previously published preliminary orbit. It was considered worth while to do part of the work which follows to six-figure accuracy. Compare the working here given with the summary of formulæ (p. 722).

			α 1925·0	δ 1925·0
I. (1)	1925 April	5·0000	336° 39′ 42″·9	+16° 30′ 8″·1
(2)		9·5000	337 47 54·4	21 17 30·8
(3)		14·0000	339 3 16·3	26 24 27·5.

	(1)	(2)	(3)	
X	+·967846	·946596	·919738	$\log \tau_1 = \log \tau_3 = 8{\cdot}888794$
Y	+·232994	·301141	·367508	$\log \tau_2$ 9·189824
Z	+·101065	·130626	·159412	τ_2^2 ·023969.

II. l	+·880362	+·862664	+·836469	$c_1^0 = c_3^0 = {\cdot}500000$
m	−·379838	−·352074	−·320177	$\nu_1 = \nu_3 = {\cdot}001498$
n	+·284053	+·363119	+·444754.	

III.	$2R_1 \cos\psi_1$	+1·584525	R_1^2 1·001226
	$2R_3 \cos\psi_3$	+1·445128	R_3^2 1·006392.

IV. m and n show the greatest change, and we therefore use their equations :

$$-{\cdot}379838c_1\rho_1 + {\cdot}352074\rho_2 - {\cdot}320177c_3\rho_3 = +{\cdot}232994c_1 - {\cdot}301141 + {\cdot}367508c_3$$
$$+{\cdot}284053c_1\rho_1 - {\cdot}363119\rho_2 + {\cdot}444754c_3\rho_3 = +{\cdot}101065c_1 - {\cdot}130626 + {\cdot}159412c_3.$$

V. $|n_2| > |m_2|$, hence we multiply the last line by $\frac{m_2}{n_2} = {\cdot}969583$ and add the result to the first, getting

$$-{\cdot}104425c_1\rho_1 + {\cdot}111049c_3\rho_3 = +{\cdot}330985c_1 - {\cdot}427794 + {\cdot}522071c_3$$
$$c_3\rho_3 = +{\cdot}940351c_1\rho_1 + 2{\cdot}980531c_1 - 3{\cdot}852300 + 4{\cdot}701267c_3.$$

From Table IV.

$$\lambda_0 : \tau_2^2 = 0 \qquad \lambda_3 : \tau_2^2 = {\cdot}25000 \quad \text{then } \lambda_3 = +{\cdot}005992,$$

and after computing M=H (in this case), N_1, N_2=0, etc., we have for our trials,

$$r_1^2 = \rho_1^2 - 1{\cdot}58452\rho_1 + 1{\cdot}00123$$
$$\rho_3 = +{\cdot}94035\rho_1 + \Delta\rho \qquad \Delta\rho = -{\cdot}022802 + {\cdot}023083\,\frac{1}{r_2^3}$$
$$r_3^2 = \rho_3^2 - 1{\cdot}44513\rho_3 + 1{\cdot}00639 \qquad s^2 = \frac{{\cdot}067808}{(r_1^2 + r_3^2)^{\frac{1}{2}}} \text{ approx.}$$

$$x_3 - x_1 = -{\cdot}093788\rho_1 + {\cdot}8365\Delta\rho + {\cdot}048108$$
$$y_3 - y_1 = +{\cdot}078759\rho_1 - {\cdot}3202\Delta\rho - {\cdot}134514$$
$$z_3 - z_1 = +{\cdot}134172\rho_1 + {\cdot}4448\Delta\rho - {\cdot}058347.$$

By the slide-rule, assuming

$$r^2=\rho^2-1\cdot515\rho+1\cdot004 \qquad \text{and (Lambert's)} \quad s^2=\cdot0479\frac{1}{r},$$

$\rho=1$	gives $r^2=0\cdot5$	$s^2=\cdot01$	s^2 (Lambert's)$=\cdot07$
$\rho=1\frac{1}{2}$	,, 0·98	·029	·049
$\rho=1\frac{3}{4}$	,, 1·41	·046	·041.

Clearly 1·7 is near the value of ρ. We therefore commence the approximations with the full equations with $\rho_1=1\cdot7000$, and with $r_2^2=r_1^2=1\cdot19755$ obtain $\Delta\rho=-\cdot00519$, $\rho_3=1\cdot59341$, $r_3^2=1\cdot24267$. Then a nearer value for $r_2^2=\frac{1}{2}(r_1^2+r_3^2)=1\cdot22011$, giving $\Delta\rho=-\cdot00567$, $\rho_3=1\cdot59293$, $r_3^2=1\cdot24183$ and leading to $s^2=\cdot041438$, s^2 (Lambert's) $=\cdot043415$, a difference of ·001977. The slide-rule computation above showed an alteration $s^2-(s^2)$ of $-\cdot025$ for a change of $\rho_1-(\rho_1)$ of ·25, hence for our difference $-\cdot002$, a correction $\partial\rho=+\cdot020$ is indicated.

VI. The corrections for the light times are introduced into the next trial. $\rho_3-\rho_1=-\cdot109$ and the correction to $\log\tau_2$, etc., is therefore $+\cdot000030$. The multiplication of N_1 and N_2 by the factor $(1+\cdot00014)$ has a negligible effect on $\Delta\rho$.

The trial is completed using $r_2=c_1{}^0r_1+c_3{}^0r_3-\cdot002410$. The result is $s^2=\cdot042816$ s^2 (Lambert's approx.)$=\cdot042816$ but $s^2=\cdot042847$ from the rigorous form of Lambert's equation. Difference $-\cdot000031$, therefore try correction $\partial\rho=+\cdot00032$. The next trial with $\rho_1=1\cdot72032$ $\Delta\rho=-\cdot00626$ $\rho_3=1\cdot61144$ $r_1=1\cdot11124$ $r_2=c_1{}^0r_1+c_3{}^0r_2-\cdot002369=1\cdot11770$ $r_3=1\cdot12889$ gives a small residual $s^2-(s^2)$ of $+\cdot000004$, which indicates the values

$$\rho_1\ 1\cdot72027 \qquad \Delta\rho\ -\cdot006268 \qquad \rho_3\ 1\cdot61139.$$

Then

η_1 ·04598	η_2 ·09236	η_3 ·04653
$\log\bar{y}_1$ ·000307	$\log\bar{y}_2$ ·001242	$\log\bar{y}_3$ ·000314,

from which

$$c_1=\cdot501078 \qquad c_3=\cdot501070.$$

The series expressions with $\frac{dr}{d\tau}=+\cdot114$ lead to $c_1=\cdot501077$, $c_3=\cdot501069$, in good agreement with the rigorous values.

A revision then gives $\lambda_0=+\cdot000022$, $\lambda_3=\cdot005976$, and these lead to $\Delta\rho=-\cdot022802+\cdot023123\frac{1}{r_2{}^3}=-\cdot006341$. The final values are then found to be

$$\rho_1\ 1\cdot72027 \qquad \rho_3\ 1\cdot61132 \qquad s^2\ \cdot042839.$$

VII.	x_1	+·546614	x_3	+·428081	yield x_2 +·488395
	y_1	−·886418	y_3	−·883416	y_2 −·886818
	z_1	+·387583	z_3	+·557229	z_2 +·473420
	r_1^2	1·23474 (check)	r_3^2	1·27418 (check).	

$r_2^2 = 1{\cdot}24910$ gives $r_2 = 1{\cdot}11763$ to check with $r_2 = 1{\cdot}11763$, the interpolated value. Also check $(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2 = s^2 = {\cdot}0428388$.

The computed middle place is then $\alpha_2 = 337^\circ\ 47'\ 51''{\cdot}7$, $\delta_2 = 21^\circ\ 17'\ 32''{\cdot}7$, and the (O−C) residuals $\partial\alpha\cos\delta + 2''{\cdot}5$, $\partial\delta - 1''{\cdot}9$, which is as good a result as can be expected.

IX. $r_1 r_3 \cos(v_3-v_1)$ 1·233043 (same from check), σ 0·998625.

$\bar{x}$	−·117781	$\bar{y}$	+·001783	$\bar{z}$	+·170179
$\bar{r}^2$	·042836$_4$	Q	·229983	$\sin(v_3-v_1)$	·183354.

The remainder of the work is done to five-figure accuracy.

X. $\log q = \log 2Q^2\left(\frac{\bar{y}_2}{2\tau_2}\right)^2 = {\cdot}04516$, but r_1, r_3, and $\frac{1}{2}(v_3-v_1) = 5^\circ{\cdot}2826$ give

$\log q$	·04513	$\frac{1}{2}v_1$	2°·230	$\frac{1}{2}v_3$	7°·513.

XI.

v_1	4°·460	v_3	15°·025		
M_1	3·2030	$\log M_3$	1·03760	$q^{\frac{3}{2}}$	1·16870,

and after correcting for the light times, t_1 −·00993, t_3 −·00930, we have

T = April 4·9901 − 3·7433 = April 13·9907 − 12·7440 = April 1·2468.

XII.

r_1	1·11119	$\sin v_1$	+·07776
$\bar{r}$	0·206970	$\cos v_1$	+·99697

yield

P_x	+·53468	P_y	−·79597	P_z	+·28380
Q_x	−·52910	Q_y	−·05344	Q_z	+·84687.

XIII.

$$x = -1{\cdot}17407\tan\frac{v}{2} + {\cdot}59323\left(1-\tan^2\frac{v}{2}\right)$$

$$y = -0{\cdot}11858\tan\frac{v}{2} - {\cdot}88313\left(1-\tan^2\frac{v}{2}\right)$$

$$z = +1{\cdot}87920\tan\frac{v}{2} + {\cdot}31488\left(1-\tan^2\frac{v}{2}\right).$$

The check, $qP_x \cdot 2qQ_x + \ldots$ gives a residual of a few units in the last place due to the accumulation of small errors.

For the middle place (using corrected date, t_2 −·00960),

$$M_2 = (\text{April } 9{\cdot}49040 - 1{\cdot}2468)q^{-\frac{3}{2}}.$$

M_2	7·0536$_5$	v_2	9° 47′·04	$\tan\frac{1}{2}v_2$	+·08559
x_2	+·48837	y_2	−·88681	z_2	+·47341,

which differ only one or two units in the last place from the values in VII.

The computed place is $\alpha_2 = 337^\circ\ 47'{\cdot}83$, $\delta_2 = +21^\circ\ 17'{\cdot}55$ and the (O−C) residuals are $\partial\alpha\cos\delta + 0'{\cdot}07$, $\partial\delta - 0'{\cdot}04$. For a cometary orbit this may be considered a satisfactory result.

XIV.

$\sin i \sin\omega$ +·57710 $\sin\Omega$ −·66898 $\cos\epsilon$ ·9174144 } 1925·0.
$\sin i \cos\omega$ +·79820 $\cos\Omega$ +·74329 $\sin\epsilon$ ·3979333 }

i being near 90°, it is best determined from the cosine, namely, by

$$\cos i = -\operatorname{cosec}\Omega\,(P_x \sin\omega + Q_x \cos\omega) = -\cdot 17265.$$

Collecting these and the other elements together, we have

T 1925 April 1·2468 G.M.T. (new).

ω 35°·8669 }
Ω 318·0119 } 1925·0 $\log q$ ·04513.
i 99·9417 }

My published orbit (*Pop. Astron.*, **33**, 335), based on the same data, was computed by method (*b*) and gave a slightly better result.

Summary of Formulæ for the Determination of a General Orbit.

I. Reduce the observed Right Ascensions and Declinations to mean places, *i.e.* referred to the mean equinox of the beginning of the year, if they are not already given as such.

α (mean place) $= \alpha$ (appt.) $-(f + \frac{1}{15}g \sin(G+\alpha)\tan\delta + \frac{1}{15}h \sin(H+\alpha)\sec\delta)$
δ (mean place) $= \delta$ (appt.) $-(g\cos(G+\alpha) + h\cos(H+\alpha))\sin\delta + i\cos\delta)$.

Reduce the observation times to G.M.T. (new) and then to decimals of a day. Interpolate the solar equatorial co-ordinates X, Y, Z (for mean equinox) for each date and time, from the *Berliner Astron. Jahrbuch* or other astronomical almanac. Compute the Local Sidereal Time, θ, for each observation, and eliminate the effect of parallax by adding in each case, to the solar co-ordinates obtained above,

$\Delta X = A\cos\theta$
$\Delta Y = A\sin\theta$
ΔZ

log A and ΔZ from Table I. (units of the seventh decimal place).
See also p. 695.

II. Compute for each of the three places:

$l = \cos\delta\cos\alpha$
$m = \cos\delta\sin\alpha$
$n = \sin\delta$

Also $\tau_1 = k(t_3 - t_2)$
$\tau_2 = k(t_3 - t_1)$
$\tau_3 = k(t_2 - t_1)$

$c_1^0 = \frac{\tau_1}{\tau_2}$ $c_3^0 = \frac{\tau_3}{\tau_2}$ check $c_1^0 + c_3^0 = 1$

$k = 0{\cdot}01720210$ $\log k = 8{\cdot}2355814_{-10}$

$$\nu_1 = \tau_2^2 a_1 + \tau_2^3 b_1 \frac{1}{r_2}\frac{dr}{d\tau}$$
$$\nu_3 = \tau_2^2 a_3 - \tau_2^3 b_3 \frac{1}{r_2}\frac{dr}{d\tau}$$

a_1, b_1, a_3, b_3 from Table II.,

or if we omit the $\frac{1}{r}\frac{dr}{d\tau}$ term (the first approximation) we can compute $\tau_2^2 a_1$ and $\tau_2^2 a_3$ as

$$\nu_1=\frac{\tau_1\tau_3}{6}(1+c_1^0) \qquad \nu_3=\frac{\tau_1\tau_3}{6}(1+c_3^0).$$

As the deduced orbit rests entirely on the fundamental data in I. and II., it is essential to check their numerical values very carefully.

III.
$$\begin{aligned} R_1\cos\psi_1&=l_1X_1+m_1Y_1+n_1Z_1 & R_1^2&=X_1^2+Y_1^2+Z_1^2\\ R_2\cos\psi_2&=l_2X_2+m_2Y_2+n_2Z_2 & R_2^2&=X_2^2+Y_2^2+Z_2^2\\ R_3\cos\psi_3&=l_3X_3+m_3Y_3+n_3Z_3 & R_3^2&=X_3^2+Y_3^2+Z_3^2. \end{aligned}$$

IV. Write down

$$\begin{aligned} l_1\cdot c_1\rho_1 -l_2\cdot\rho_2 +l_3\cdot c_3\rho_3 &=X_1c_1-X_2+X_3c_3\\ m_1\cdot c_1\rho_1-m_2\cdot\rho_2+m_3\cdot c_3\rho_3&=Y_1c_1-Y_2+Y_3c_3\\ n_1\cdot c_1\rho_1-n_2\cdot\rho_2+n_3\cdot c_3\rho_3&=Z_1c_1-Z_2+Z_3c_3, \end{aligned}$$

putting in the numerical values of all the quantities except $c_1\rho_1$, $c_3\rho_3$, c_1, and c_3. Eliminate numerically, in the ordinary way, the quantities $c_1\rho_1$ and $c_3\rho_3$, and thus obtain an equation of the form

V. $$E\rho_2=F_1c_1+F_2+F_3c_3.$$

Compute A and B from

$$EA=c_1^0F_1+F_2+c_3^0F_3 \quad \text{and} \quad EB=\nu_1F_1+\nu_3F_3.$$

For the special case where E=0, see p. 697. Then solve the fundamental equations

$$\rho_2=A+\frac{B}{r_2^3} \qquad \left(\text{Note} \quad A=-\frac{B}{R_2^3} \text{ nearly}\right).$$

$$\begin{aligned} r_2^2&=\rho_2^2-2R_2\cos\psi_2\cdot\rho_2+R_2^2\\ &=(\rho_2-R_2\cos\psi_2)^2+R_2^2\sin^2\psi_2 \qquad R_2^2\sin^2\psi_2=R_2^2-R_2^2\cos^2\psi_2. \end{aligned}$$

With the values of ρ_2 and r_2 thus found, compute

$$c_1=c_1^0+\frac{\nu_1}{r_2^3} \qquad c_3=c_3^0+\frac{\nu_3}{r_2^3},$$

and using these values in the equations intermediate between IV. and V., and lastly in the equations IV. (use all three as check), obtain $c_1\rho_1$ and $c_3\rho_3$, and hence ρ_1 and ρ_3 (provisional values). Also

$$r_1^2=\rho_1^2-2R_1\cos\psi_1\cdot\rho_1+R_1^2 \qquad r_3^2=\rho_3^2-2R_3\cos\psi_3\cdot\rho_3+R_3^2.$$

VI. The observation dates may now be corrected for the light times. In each case

$$t\ (\text{corrected})=t-\cdot005770\rho\ (\text{days}) \qquad \log\cdot005770=7\cdot7612,$$

and recompute c_1^0, c_3^0, ν_1, and ν_3 in II.; or it is usually sufficient to

multiply the values of ν_1 and ν_3 originally found, also B in V., by the factor

$$\frac{\tau_2^2 \text{ (corrected)}}{\tau_2^2 \text{ (uncorrected)}} = 1 - \cdot 01154 \frac{\rho_3 - \rho_1}{t_3 - t_1}.$$

Correct ν_1 and ν_3 to include also the $\frac{1}{r}\frac{dr}{d\tau}$ term in II., using an interpolated value of $\frac{dr}{d\tau}$, *e.g.* $\frac{dr}{d\tau} = \frac{r_3 - r_1}{\tau_2}$.

Alternatively, with the corrected time intervals, etc., and r_1, r_2, r_3, compute Gibbs's expressions

$$\mu_1 = \tfrac{1}{12}(\tau_2\tau_3 - \tau_1^2); \quad \mu_2 = \tfrac{1}{12}(\tau_1\tau_3 + \tau_2^2); \quad \mu_3 = \tfrac{1}{12}(\tau_1\tau_2 - \tau_3^2),$$

or take $\mu_1 : \tau_2^2$, $\mu_2 : \tau_2^2$, $\mu_3 : \tau_2^2$ from Table III., and hence obtain μ_1, μ_2, μ_3.

Then

$$c_1 = c_1^0 \frac{1 + \frac{\mu_1}{r_1^3}}{1 - \frac{\mu_2}{r_2^3}} \qquad c_3 = c_3^0 \frac{1 + \frac{\mu_3}{r_3^3}}{1 - \frac{\mu_2}{r_2^3}},$$

and *corrected*

$$\nu_1 = (c_1 - c_1^0) r_2^3 \qquad \nu_3 = (c_3 - c_3^0) r_2^3.$$

For long arcs where these expressions are not adequate (see p. 699) proceed straight to VII. for the computation of x_1, y_1, z_1, x_3, y_3, z_3, and to VIII. to obtain c_1 and c_3 from $\bar{y}_1$, $\bar{y}_2$, and $\bar{y}_3$.

As much of the original solution must be revised as is appreciably affected by these corrections, but it will often be found that only ρ_1 and ρ_3 need to be recomputed with the revised values of c_1 and c_3.

VII. $x_1 = \rho_1 l_1 - X_1$ $\quad x_3 = \rho_3 l_3 - X_3$ Checks $r_1^2 = x_1^2 + y_1^2 + z_1^2$
$y_1 = \rho_1 m_1 - Y_1$ $\quad y_3 = \rho_3 m_3 - Y_3$ $\quad r_3^2 = x_3^2 + y_3^2 + z_3^2$
$z_1 = \rho_1 n_1 - Z_1$ $\quad z_3 = \rho_3 n_3 - Z_3$ (similarly, x_2, y_2, z_2, from ρ_2).

Then for a check on the middle place we have

$x_2 = c_1 x_1 + c_3 x_3$ $\quad \rho_2 \cos\delta_2 \cos\alpha_2 = x_2 + X_2$ $\quad r_2^2 = x_2^2 + y_2^2 + z_2^2$
$y_2 = c_1 y_1 + c_3 y_3$ $\quad \rho_2 \cos\delta_2 \sin\alpha_2 = y_2 + Y_2$
$z_2 = c_1 z_1 + c_3 z_3$ $\quad \rho_2 \sin\delta_2 = z_2 + Z_2$

Check with previous ρ_2 and r_2, and observed values of α_2 and δ_2.

It is usually more convenient to compute this check when the final values of c_1 and c_3 are available from VIII.

VIII. Using corrected τ_1, τ_2, τ_3 we compute $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, the ratios of the corresponding sectors : triangles.* For $(v_3 - v_1)$ up to 20° (seven-figure work) or 30° (six-figure work) we have

* For the first method (Hansen's) see C. Veithen, *loc. cit.*, p. 13. See also various textbooks and in particular J. Bauschinger, *Bahnbestimmung*, p. 154 *et seq.*, and *Tafeln*, p. 24, etc., where several different methods, with and without tables, are detailed.

$$r_1 r_3 \cos(v_3 - v_1) = x_1 x_3 + y_1 y_3 + z_1 z_3 \qquad \kappa_2{}^2 = r_1 r_3 + r_1 r_3 \cos(v_3 - v_1)$$

$$h_2 = -\frac{{\tau_2}^2}{\kappa^2\left(\frac{2\sqrt{2}}{3}\kappa + r_1 + r_3\right)} \qquad \text{then} \qquad \bar{y}_2 = 1 + \frac{10}{11}\cdot\cfrac{\frac{11}{9}h_2}{1+\cfrac{\frac{11}{9}h_2}{1+\cfrac{\frac{11}{9}h_2}{1+\ldots}}}$$

$$\frac{2\sqrt{2}}{3} = \cdot 9428090 \qquad \log = 9\cdot 9744237.$$

Similarly we compute $\bar{y}_1$ from $\tau_1 r_2 r_3$ and $r_2 r_3 \cos(v_3 - v_2) = x_2 x_3 + y_2 y_3 + z_2 z_3$

" " " $\bar{y}_3$ from $\tau_3 r_1 r_2$ and $r_1 r_2 \cos(v_2 - v_1) = x_1 x_2 + y_1 y_2 + z_1 z_2$

Several other alternative methods are available for finding $\bar{y}_1, \bar{y}_2, \bar{y}_3$; for example, the following convenient formula by Encke, which is suitable in the case of minor planets up to intervals of 60 days or more. $\cos(v_3 - v_1)$, above, will give us $(v_3 - v_1)$ with sufficient accuracy to get $\cos\frac{1}{2}(v_3 - v_1)$, then

$$\cos\gamma_2 = \frac{2\sqrt{r_1 r_3}\cos\frac{1}{2}(v_3 - v_1)}{r_1 + r_3} \qquad \Gamma = \log\frac{1}{2\sin^2\frac{1}{2}\gamma_2}.$$

Γ_2 can be taken from the Zech subtraction logarithms with argument $\log\sec\gamma_2$

$$\bar{\eta}_2 = \frac{{\tau_2}^2}{k^2(r_1 + r_3)^3}$$

$$\log k^2 = 6\cdot 4711629_{-10}$$
$$\log a' = 3\cdot 2338859$$
$$\log a'' = 3\cdot 6140972 - \Gamma_2$$
$$\log b'' = 0\cdot 0341076$$

$$\log\bar{y}_2 = a'\bar{\eta}_2 + a''\bar{\eta}_2 - b''\bar{\eta}_2{}^2 \qquad \text{(in units of the seventh decimal place).}$$

Similarly $\log\bar{y}_1$ and $\log\bar{y}_3$.

For other methods, and where longer arcs are used, see the formulæ in standard works.*

Then,

$$\textit{corrected} \quad c_1 = c_1{}^0\frac{\bar{y}_2}{\bar{y}_1} \qquad c_3 = c_3{}^0\frac{\bar{y}_2}{\bar{y}_3}.$$

Compare these with the previous values found and revise the solution if there is any appreciable difference.

Note,

$$\textit{corrected} \quad \nu_1 = c_1{}^0\left(\frac{\bar{y}_2}{\bar{y}_1} - 1\right){r_2}^3 \qquad \nu_3 = c_3{}^0\left(\frac{\bar{y}_2}{\bar{y}_3} - 1\right){r_2}^3.$$

IX. $\sigma = \dfrac{r_1 r_3 \cos(v_3 - v_1)}{{r_1}^2}$ Make $\bar{x}, \bar{y}, \bar{z}$ up to the (six-figure) accuracy required.

$\bar{x} \quad x_3 - \sigma x_1$ $\qquad\qquad \bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$

$\bar{y} \quad y_3 - \sigma y_1$ $\qquad\qquad Q = r_1 r_3 \sin(v_3 - v_1) - r_1\bar{r} \qquad (\textit{positive})$

$\bar{z} \quad z_3 - \sigma z_1$ $\qquad$ check Q^2 $\quad (y_1 z_3 - y_3 z_1)^2 + (x_1 z_3 - x_3 z_1)^2 + (x_1 y_3 - x_3 y_1)^2$.

* For instance, J. Bauschinger, *Bahnbestimmung*, p. 154 *et seq.*, and *Tafeln*, p. 24, etc.

For the purpose of checks, here and in X, also compute if desired,*

$$r_1r_2 \sin (v_2-v_1)=\{(y_1z_2-y_2z_1)^2+(x_1z_2-x_2z_1)^2+(x_1y_2-x_2y_1)^2\}^{\frac{1}{2}}$$
$$r_2r_3 \sin (v_3-v_2)=\{(y_2z_3-y_3z_2)^2+(x_2z_3-x_3z_2)^2+(x_2y_3-x_3y_2)^2\}^{\frac{1}{2}}.$$

Then as checks,

$$c_1=\frac{r_2r_3 \sin (v_3-v_2)}{r_1r_3 \sin (v_3-v_1)} \qquad c_3=\frac{r_1r_2 \sin (v_2-v_1)}{r_1r_3 \sin (v_3-v_1)}.$$

X. $\sqrt{p}=\frac{r_1\bar{r}}{\tau_2}\bar{y}_2$ as checks, $\sqrt{p}=\frac{r_2r_3 \sin (v_3-v_2)}{\tau_1}\bar{y}_1=\frac{r_1r_2 \sin (v_2-v_1)}{\tau_3}\bar{y}_3.$

Put

$$q_1=\frac{p}{r_1}-1 \qquad q_3=\frac{p}{r_3}-1.$$

Then

$$e \cos v_1=q_1$$
$$e \sin v_1=\frac{q_1 \cos (v_3-v_1)-q_3}{\sin (v_3-v_1)} \qquad e=\sin \phi \qquad a=p \sec^2 \phi.$$

Hence, knowing (v_2-v_1) and (v_3-v_2), find also v_2 and v_3. Checks,

$$p=r_2(1+e \cos v_2)=r_3(1+e \cos v_3).$$

XI. $\sin \frac{1}{2}(v_1-E_1)=\sqrt{\frac{r_1}{p}} \sin \frac{1}{2}\phi \sin v_1$ or $\tan \frac{1}{2}E_1=\sqrt{\frac{1-e}{1+e}} \tan \frac{1}{2}v_1$

$\sin \frac{1}{2}(v_3-E_3)=\sqrt{\frac{r_3}{p}} \sin \frac{1}{2}\phi \sin v_3$ $\qquad$ $\tan \frac{1}{2}E_3=\sqrt{\frac{1-e}{1+e}} \tan \frac{1}{2}v_3.$

Hence

E_1 and E_3. Check, $a \cos \phi \sin \frac{1}{2}(E_3-E_1)=\sqrt{r_1r_3} \sin \frac{1}{2}(v_3-v_1).$

Then (in degrees)

$M_1=E_1-e^\circ \sin E_1$ $\qquad$ $M_3=E_3-e^\circ \sin E_3$ $\qquad$ $e^\circ=57{\cdot}29578e,$

and using the *corrected* dates t_3, t_1, $\qquad$ $\log (e^\circ : e)=1{\cdot}7581226.$

The mean motion

$$\mu^\circ=\frac{k^\circ}{a^{\frac{3}{2}}}=\frac{M_3-M_1}{t_3-t_1} \text{ (check)} \qquad \log k^\circ=9{\cdot}9937041_{-10}$$
$$\text{for } \mu'', \log k''=3{\cdot}5500066.$$

XII.
$$P_x=x_1\frac{\cos v_1}{r_1}-\bar{x}\frac{\sin v_1}{\bar{r}} \qquad Q_x=x_1\frac{\sin v_1}{r_1}+\bar{x}\frac{\cos v_1}{\bar{r}}$$
$$P_y=y_1\frac{\cos v_1}{r_1}-\bar{y}\frac{\sin v_1}{\bar{r}} \qquad Q_y=y_1\frac{\sin v_1}{r_1}+\bar{y}\frac{\cos v_1}{\bar{r}}$$
$$P_z=z_1\frac{\cos v_1}{r_1}-\bar{z}\frac{\sin v_1}{\bar{r}} \qquad Q_z=z_1\frac{\sin v_1}{r_1}+\bar{z}\frac{\cos v_1}{\bar{r}}.$$

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* The computation of these and the subsequent checks depending on them may be omitted, provided Q is checked and the checks in XI. are worked.

Checks,

$$P_x^2+P_y^2+P_z^2=1, \quad Q_x^2+Q_y^2+Q_z^2=1, \quad P_xQ_x+P_yQ_y+P_zQ_z=0.$$

XIII. For an ephemeris use

$$x=aP_x(\cos E-e)+bQ_x \sin E \qquad b=a\cos\phi \qquad r=a(1-e\cos E)$$
$$y=aP_y(\cos E-e)+bQ_y \sin E$$
$$z=aP_z(\cos E-e)+bQ_z \sin E$$

Check,

$$aP_x \,.\, bQ_x+aP_y \,.\, bQ_y+aP_z \,.\, bQ_z=0,$$

where E is obtained from

$$M_1+\mu(t-t_1)=E-e^\circ \sin E,$$

and

$$\rho\cos\delta\cos\alpha=x+X \qquad \rho\cos\delta\sin\alpha=y+Y \qquad \rho\sin\delta=z+Z.$$

As a check, compute the middle place by these equations.

XIV. Find the ecliptic elements ω, Ω, i from

$$\sin i\sin\omega=P_z\cos\epsilon-P_y\sin\epsilon$$
$$\sin i\cos\omega=Q_z\cos\epsilon-Q_y\sin\epsilon$$

ϵ=mean obliquity of ecliptic (for the beginning of the year). $\sin i$, *positive.*

$$\sin\Omega=(P_y\cos\omega-Q_y\sin\omega)\sec\epsilon$$
$$\cos\Omega=P_x\cos\omega-Q_x\sin\omega.$$

As a check, note that $\sin\Omega$ and $\cos\Omega$ should agree *inter se.*

The following also checks the quadrant of i:

$$P_x\sin\omega+Q_x\cos\omega=-\cos i\sin\Omega.$$

Formulæ for the Determination of a Parabolic Orbit, Method (a).

I., II., III., and as far as the equations in IV., are the same as in the General Orbit, except that we select the two equations showing the greatest motion in l, m, or n, and do not compute $R_2\cos\psi_2$.

V. Eliminate ρ_2 from the two selected equations in IV. and obtain the equation of the form

$$c_3\rho_3=Hc_1\rho_1+F_1c_1+F_2+F_3c_3.$$

With argument $c_1{}^0$ take $\lambda_0:\tau_2{}^2$ and $\lambda_3:\tau_2{}^2$ from Table IV., and compute λ_0 and λ_3, then

$$M=\frac{c_1{}^0}{c_3{}^0}H \qquad N_1=\lambda_0F_1-\lambda_3F_2$$

$$N_0=\frac{c_1{}^0}{c_3{}^0}(F_1+F_2)+F_2+F_3 \qquad N_2=\lambda_0H.$$

For the solution, commence with a trial value of ρ_1, then

$$r_1{}^2=\rho_1{}^2-2R_1\cos\psi_1\,.\,\rho_1+R_1{}^2$$

$$\rho_3=M\rho_1+\Delta\rho \quad \text{where } \Delta\rho=N_0+(N_1+N_2\rho_1)\frac{1}{r_2{}^3},$$

and for this we improve the approximation for r_2 at each step by

$$r_2 = c_1{}^0 r_1 + c_3{}^0 r_3 - c_1{}^0 c_3{}^0 \frac{\tau_2{}^2 - r_2(r_3 - r_1)^2}{2 r_2{}^2} \quad \text{(see p. 704)}$$

$$r_3{}^2 = \rho_3{}^2 - 2\mathrm{R}_3 \cos\psi_3 \cdot \rho_3 + \mathrm{R}_3{}^2$$

$$\eta = \frac{2\tau_2}{(r_1+r_3)^{\frac{3}{2}}} \qquad s = \eta\mu(r_1+r_3) = \mu\frac{2\tau_2}{(r_1+r_3)^{\frac{1}{2}}}.$$

μ is taken from the table for this purpose with argument η.*

A near approximation † for short arcs is

$$s^2 = 2{\cdot}829\frac{\tau_2{}^2}{(r_1{}^2+r_3{}^2)^{\frac{1}{2}}} \qquad \log 2{\cdot}829 = {\cdot}4516.$$

Then

$$\begin{aligned} x_3 - x_1 &= \rho_1(\mathrm{M}l_3 - l_1) \quad + \Delta\rho \cdot l_3 \; + (\mathrm{X}_1 - \mathrm{X}_3) \\ y_3 - y_1 &= \rho_1(\mathrm{M}m_3 - m_1) + \Delta\rho \cdot m_3 + (\mathrm{Y}_1 - \mathrm{Y}_3) \\ z_3 - z_1 &= \rho_1(\mathrm{M}n_3 - n_1) \; + \Delta\rho \cdot n_3 + (\mathrm{Z}_1 - \mathrm{Z}_3) \\ s^2 &= (x_3-x_1)^2 + (y_3-y_1)^2 + (z_3-z_1)^2. \end{aligned}$$

When the $\frac{1}{r_2{}^3}$ term in $\Delta\rho$ is known with sufficient accuracy, it may be convenient to obtain the usual form of equation,

$$s^2 = \rho_1{}^2\mathrm{P}_1 + \rho_1\mathrm{P}_2 + \mathrm{P}_3,$$

for getting s^2 directly from ρ_1 in each trial.

Approximations are continued till the two values of s obtained agree. Five-figure work for the chord and the remainder of the working is usually all that is justified for cometary orbits.

VI. When approximate values of the geocentric distances have been found, correct for the light times before proceeding further. With sufficient accuracy add correction to $\log 2\tau_2 = -\frac{1}{400}\frac{\rho_3 - \rho_1}{t_3 - t_1}$, and the same correction to $\log \tau_1$ and $\log \tau_3$.

Multiply N_1 and N_2 (also ν_1 and ν_3) by

$$\frac{\tau_2{}^2 \text{ (corrected)}}{\tau_2{}^2 \text{ (uncorrected)}} = 1 - {\cdot}01154\frac{\rho_3 - \rho_1}{t_3 - t_1}.$$

Note for subsequent use

$$t \text{ (corrected)} = t - {\cdot}005770\rho \qquad \log \text{factor} = 7{\cdot}7612.$$

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* See references p. 703(*), but if tables are not available, compute this μ from

$$\mu = 1 + \tfrac{1}{2}\tfrac{1}{4}\eta^2 + \tfrac{5}{8}\tfrac{5}{4}\eta^4 + \tfrac{5}{8}\tfrac{5}{2}\tfrac{9}{16}\eta^6 + \ldots$$

or use equation (53), p. 702.

† Due to B. Strömgren, *A.N.*, **224**, 343, who gives it as $s^2 = 0{\cdot}0008370\frac{(t_3-t_1)^2}{(r_1{}^2+r_3{}^2)^{\frac{1}{2}}}$. The chief error then arises from putting $r_1 + r_3 = \sqrt{2(r_1{}^2+r_3{}^2)}$.

With the quantities thus corrected complete the solution in V. Then for the values of c_1 and c_3 compute

$$c_1=c_1^0+\frac{v_1}{r_2^3} \qquad c_3=c_3^0+\frac{v_3}{r_2^3},$$

but for the correction of these quantities, and λ_0 and λ_3, instead of proceeding as in VI. for the general orbit, go direct to the $\bar{y}$ expressions (as in VIII.). Use the interpolated value of r_2 and corrected time intervals, and with

$$\eta_1=\frac{2\tau_1}{(r_2+r_3)^{\frac{3}{2}}} \qquad \eta_2=\frac{2\tau_2}{(r_1+r_3)^{\frac{3}{2}}} \qquad \eta_3=\frac{2\tau_3}{(r_1+r_2)^{\frac{3}{2}}}$$

as arguments, take $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ from the table for this purpose,* or in each case compute them from

$$\sin\gamma=\eta\mu \quad \text{then } \bar{y}=\frac{1+2\sec\gamma}{3}.$$

Then,

$$\textit{corrected} \quad c_1=c_1^0\frac{\bar{y}_2}{\bar{y}_1} \qquad c_3=c_3^0\frac{\bar{y}_2}{\bar{y}_3}.$$

Compare these with the values obtained above. Also,

$$\textit{corrected} \quad \lambda_0=\frac{c_1^0}{c_3^0}\left(\frac{\bar{y}_3}{\bar{y}_1}-1\right)r_2^3 \qquad \lambda_3=\left(\frac{c_1^0}{c_3^0}+1\right)\left(\frac{\bar{y}_2}{\bar{y}_3}-1\right)r_2^3.$$

If necessary, the solution should be revised with these values.

VII. The same as in the General Orbit. Compute the middle place to find the residuals. If these are too large, another solution is indicated, or a general solution is required (see p. 705).

VIII. c_1 and c_3 have already been obtained from $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ in VI.

IX. As in the General Orbit, but omit the computation of $r_1r_2\sin(v_2-v_1)$ and $r_2r_3\sin(v_3-v_2)$, and note

$$r_1r_3\cos(v_3-v_1)=x_1x_3+y_1y_3+z_1z_3$$
$$=\tfrac{1}{2}(r_1^2+r_3^2-s^2) \quad \text{(check)}.$$

X. $$\frac{1}{\sqrt{q}}\cos\tfrac{1}{2}v_1=\frac{1}{\sqrt{r_1}}$$

$$\frac{1}{\sqrt{q}}\sin\tfrac{1}{2}v_1=\frac{1}{\sqrt{r_1}}\cot\tfrac{1}{2}(v_3-v_1)-\frac{1}{\sqrt{r_3}}\operatorname{cosec}\tfrac{1}{2}(v_3-v_1)$$

or †

$$\tan\tfrac{1}{2}(v_1+v_3)=\frac{(r_3-r_1)\sin\tfrac{1}{2}(v_3-v_1)}{2\sqrt{r_1r_3}-(r_1+r_3)\cos\tfrac{1}{2}(v_3-v_1)},$$

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* See references p. 700(*).

† Th. Banachiewicz, *Circ. de l'Observ. de Cracovie*, **16**, 3.

where $\frac{1}{2}(v_1+v_3)$ has the sign of (r_3-r_1), and we know (v_1-v_3) from IX., and get the perihelion distance q from

$$q=r_1\cos^2\frac{v_1}{2}=r_3\cos^2\frac{v_3}{2}, \quad \text{(check)}.$$

or, knowing

$$\left(\log\cos\frac{v_3}{2}-\log\cos\frac{v_1}{2}\right)=\tfrac{1}{2}\log\frac{r_1}{r_3} \quad \text{and} \quad \left(\frac{v_3}{2}-\frac{v_1}{2}\right) \quad \text{find} \quad \frac{v_3}{2} \text{ and } \frac{v_1}{2}$$

by inspection from log. trig. tables, then as before

$$q=r_1\cos^2\frac{v_1}{2}=r_3\cos^2\frac{v_3}{2}, \quad \text{(check)}.$$

Check: With $\bar{y}_2$ from VI. we have q in the parabola directly from

$$q=2\mathrm{Q}^2\left(\frac{\bar{y}_2}{2\tau_2}\right)^2.$$

XI. With arguments v_1 and v_3 take M_1 and M_3 from the table for this purpose. Then the perihelion date

$$\mathrm{T}=t_1-\frac{\mathrm{M}_1}{\mathrm{C}}q^{\frac{3}{2}}=t_3-\frac{\mathrm{M}_3}{\mathrm{C}}q^{\frac{3}{2}} \quad \text{(using } \textit{corrected} \text{ dates } t_1,\ t_3).$$

Bauschinger, Table XV.; Wirtz, Table 50*a*, use C=1.
Watson, *Theor. Astron.*, Table VI., uses $\log \mathrm{C}=9{\cdot}9601277_{-10}$,

or from

$$t-\mathrm{T}=\mathrm{M}q^{\frac{3}{2}}=\frac{\sqrt{2}}{k}\left(\tan\frac{v}{2}+\tfrac{1}{3}\tan^3\frac{v}{2}\right)q^{\frac{3}{2}},$$

in each case.

$$\frac{\sqrt{2}}{k}=82{\cdot}2117 \qquad \log\frac{\sqrt{2}}{k}=1{\cdot}914934.$$

XII. As in the General Orbit.

XIII.
$$x=q\mathrm{P}_x\left(1-\tan^2\frac{v}{2}\right)+2q\mathrm{Q}_x\tan\frac{v}{2}$$
$$y=q\mathrm{P}_y\left(1-\tan^2\frac{v}{2}\right)+2q\mathrm{Q}_y\tan\frac{v}{2}$$
$$z=q\mathrm{P}_z\left(1-\tan^2\frac{v}{2}\right)+2q\mathrm{Q}_z\tan\frac{v}{2},$$

Check,
$$q\mathrm{P}_x{}^2q\mathrm{Q}_x+q\mathrm{P}_y{}^2\mathrm{Q}_y+q\mathrm{P}_z{}^2q\mathrm{Q}_z=0$$

where v is obtained from the table (see XI.),* with argument

$$\mathrm{M}=\mathrm{C}(t-\mathrm{T})q^{-\frac{3}{2}}, \quad \text{and as usual,}$$

$$\rho\cos\delta\cos\alpha=x+\mathrm{X} \qquad \rho\cos\delta\sin\alpha=y+\mathrm{Y} \qquad \rho\sin\delta=z+\mathrm{Z}.$$

* For the solution of the cubic to find v, when tables are not available, see, for example, E. O. Fountain, *B.A.A. Journal*, **31**, 196; Miss M. C. Traylor, *Pop. Astron.*, **9**, 339.

As a check compute the middle place by these equations.

XIV. As in the General Orbit.

Formulæ for the Determination of a Parabolic Orbit, Method (b).

This method is to be used when r_2 cannot be interpolated easily to the required accuracy.

I. to IV. are as before, but compute $R_2 \cos \psi_2$ and also eliminate $c_1\rho_1$ (or $c_3\rho_3$) from the two selected equations and obtain the equation of the form

V. $$c_3\rho_3 = H\rho_2 + F_1c_1 + F_2 + F_3c_3.$$

Compute

$$J_0 = c_1{}^0F_1 + F_2 + c_3{}^0F_3 \qquad J_2 = \nu_1F_1 + \nu_3F_3.$$

For the solution, commence with a trial value of ρ_2, then

$$r_2{}^2 = \rho_2{}^2 - 2R_2 \cos \psi_2 \,.\, \rho_2 + R_2{}^2$$

$$c_3 = c_3{}^0 + \frac{\nu_3}{r_2{}^3}.$$

From

$$c_2\rho_3 = H\rho_2 + J_0 + \frac{J_2}{r_2{}^3} \qquad \text{obtain } \rho_3, \text{ then} \qquad \Delta\rho = \rho_3 - \rho_2$$

$$\begin{aligned} r_3{}^2 &= \rho_3{}^2 - 2R_3 \cos \psi_3 \,.\, \rho_3 + R_3{}^2 \\ x_3 - x_2 &= \rho_2(l_3 - l_2) + \Delta\rho \,.\, l_3 + (X_2 - X_3) \\ y_3 - y_2 &= \rho_2(m_3 - m_2) + \Delta\rho \,.\, m_3 + (Y_2 - Y_3) \\ z_3 - z_2 &= \rho_2(n_3 - n_2) + \Delta\rho \,.\, n_3 + (Z_2 - Z_3) \\ s^2 &= (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 \end{aligned}$$

$$\eta = \frac{2\tau_1}{(r_2 + r_3)^{\frac{1}{2}}} \qquad s = \eta\mu(r_2 + r_3) = \mu\frac{2\tau_1}{(r_2 + r_3)^{\frac{1}{2}}}. \qquad \text{See ref. p. 723(*).}$$

Approximations are continued till the two values of s agree.

VI. When approximate values of the geocentric distances have been found, correct for the light times before proceeding further. With sufficient accuracy, usually, add correction to $\log 2\tau_1 = -\frac{1}{400}\frac{\rho_3 - \rho_2}{t_3 - t_2}$, and the same correction to $\log \tau_2$ and $\log \tau_3$, and multiply ν_1, ν_3, and J_2 by

$$\frac{\tau_2{}^2 \text{ (corrected)}}{\tau_2{}^2 \text{ (uncorrected)}} = 1 - \cdot 01154\frac{\rho_3 - \rho_2}{t_3 - t_2}.$$

Complete the solution with these corrected quantities and then compute c_1 and c_3 as before, also $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, etc., for which use an extrapolated value of r_1.

If c_1 and c_3 are appreciably altered, revise the solution with

corrected $$\nu_1 = c_1{}^0\left(\frac{\bar{y}_2}{\bar{y}_1} - 1\right)r_2{}^3 \qquad \nu_3 = c_3{}^0\left(\frac{\bar{y}_2}{\bar{y}_3} - 1\right)r_2{}^3.$$

VII. $x_2=\rho_2 l_2 - X_2$ $\quad x_3=\rho_3 l_3 - X_3$ $\quad$ Checks,
$y_2=\rho_2 m_2 - Y_2$ $\quad y_3=\rho_3 m_3 - Y_3$ $\quad r_2^2=x_2^2+y_2^2+z_2^2$
$z_2=\rho_2 n_2 - Z_2$ $\quad z_3=\rho_3 n_3 - Z_3$ $\quad r_3^2=x_3^2+y_3^2+z_3^2$.

Then for a check on the first place,

$c_1 x_1=x_2-c_3 x_3$ $\quad \rho_1 \cos\delta_1 \cos\alpha_1=x_1+X_1$ $\quad r_1^2=x_1^2+y_1^2+z_1^2$
$c_1 y_1=y_2-c_3 y_3$ $\quad \rho_1 \cos\delta_1 \sin\alpha_1=y_1+Y_1$
$c_1 z_1=z_2-c_3 z_3$ $\quad \rho_1 \sin\delta_1 \quad =z_1+Z_1$.

With the value of ρ_1 so obtained compute x_1, y_1, z_1 from the *observed* α_1 and δ_1 by

$$x_1=\rho_1 l_1 - X_1 \qquad y_1=\rho_1 m_1 - Y_1 \qquad z_1=\rho_1 n_1 - Z_1$$

and retaining x_3, y_3, and z_3 unaltered, modify ρ_1 till the value of the chord s from

$$s^2=(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2$$

agrees with the value from

$$\eta=\frac{2\tau_2}{(r_1+r_3)^{\frac{3}{2}}} \qquad s=\eta\mu(r_1+r_3)=\mu\frac{2\tau_2}{(r_1+r_3)^{\frac{1}{2}}}.$$

The adjustment required is usually very small, but if not small it may be necessary to adjust ρ_3, using the first equation in V. of method (*a*)—see also remarks to VII. of method (*a*).

The remainder of the work, from IX. onwards is carried through as in method (*a*).

TABLE I.

Parallax Constants.

(A and ΔZ in units of the 7th decimal place.)

Observatory.	Longitude W+*ve.* h m	log A.	ΔZ.
Algiers	-0 12·1	2_n534	−254
Arequipa	+4 46·2	2_n612	+102
Athens	−1 34·9	2_n527	−261
Bamberg	-0 43·6	2_n440	−325
Barcelona	−0 8·6	2_n506	−281
Bergedorf (Hamburg)	-0 41·0	2_n406	−341
Berlin-Babelsberg	-0 52·4	2_n416	−337
Besançon	−0 24·0	2_n463	−312
Bonn	-0 28·4	2_n432	−329
Bordeaux	-0 2·1	2_n482	−299
Cape	-1 13·9	2_n550	+237
Copenhagen	-0 50·3	2_n382	−351
Düsseldorf	−0 27·0	2_n428	−331
Evanston (Dearborn Observatory)	+5 50·7	2_n501	−284
Frankfurt a.M.	-0 34·6	2_n438	−326
Göttingen	-0 39·8	2_n425	−333
Greenwich	0 0·0	2_n425	−332
Harvard	+4 44·5	2_n499	−286
Jena	−0 46·3	2_n431	−330
Johannesburg	−1 52·3	2_n583	+187
Kasan (Univ. Obs.)	−3 16·5	2_n381	−351
Königsberg	−1 22·0	2_n393	−347
Königstuhl	−0 34·9	2_n444	−322
La Plata	+3 51·7	2_n544	+243
Lick	+8 6·6	2_n531	−257
Lisbon	+0 36·7	2_n523	−265
Marseille	−0 21·6	2_n493	−24[illegible]
Melbourne	−9 39·9	2_n528	+2[illegible]
Milan	−0 36·8	2_n477	−3[illegible]
Minneapolis	+6 13·0	2_n481	−3[illegible]
Munich	−0 46·4	2_n455	−3[illegible]
Nice	−0 29·2	2_n490	−2[illegible]
Northfield (Goodsell Observatory)	+6 12·6	2_n484	−2[illegible]
Padua	−0 47·5	2_n477	−3[illegible]
Paris	−0 9·3	2_n449	−32[illegible]
Pulkowo	−2 1·3	2_n333	−3[illegible]
Rome (Coll. Rom.)	−0 49·9	2_n503	−28[illegible]
San Fernando	-1 0 24·8	2_n536	−2[illegible]
Santiago	+4 42·8	2_n552	+2[illegible]
Simeïs	−2 16·0	2_n485	−2[illegible]
Sonneberg	−0 44·7	2_n436	−32[illegible]
Stockholm	−1 12·2	2_n339	−3[illegible]
Strasbourg	−0 31·1	2_n452	−31[illegible]
Swarthmore (Sproul Observatory)	+5 1·4	2_n516	−27[illegible]
Tacubaya	+6 36·8	2_n605	−14[illegible]
Uccle	−0 17·4	2_n432	−32[illegible]
Utrecht	−0 20·5	2_n420	−33[illegible]
Vienna	−1 5·4	2_n455	−31[illegible]
Washington (Naval Observatory)	+5 8·3	2_n522	−26[illegible]
Yerkes	+5 54·2	2_n498	−28[illegible]

A is always −*ve*. ΔZ has the *opposite* sign to ϕ (Geogr. latitude).

ϕ+.	log A.	ΔZ.	ϕ+.	log A.	ΔZ.	ϕ+.	log A.	ΔZ.
°			°			°		
0	2_n630	− 0	40	2_n515	−273	56	2_n379	−352
	2 37			13 11			21 8	
5	2_n628	37	42	2_n502	−284	58	2_n355	−360
	5 37			14 11			25 8	
10	2_n623	74	44	2_n488	−295	60	2_n330	−368
	8 36			16 10			27 7	
15	2_n615	−110	46	2_n472	−305	62	2_n303	−375
	12 35			16 11			30 7	
20	2_n603	−145	48	2_n456	−316	64	2_n273	−382
	15 31			17 9			32 6	
25	2_n588	179	50	2_n439	−325	66	2_n241	−388
	20 33			19 10			36 6	
30	2_n568	212	52	2_n420	−335	68	2_n205	394
	24 31			20 9			40 5	
35	2_n544	243	54	2_n400	−344	70	2_n165	−399
	29 30			21 8				

Table II.

$c_1°$.	a_1.		b_1.		a_3.		b_3.		$c_3°$.
0·20	+·03200		+·0304		+·04800		−·0176		0·80
		146		5		149		4	
·21	·03346		·0309		·04949		·0172		·79
		143		5		142		5	
·22	·03489		·0314		·05091		·0167		·78
		142		4		133		6	
·23	·03631		·0318		·05224		·0161		·77
		139		2		126		7	
·24	·03770		·0320		·05350		·0154		·76
		136		2		119		8	
·25	·03906		·0322		·05469		−·0146		·75
		134		1		111		8	
·26	·04040		·0323		·05580		·0138		·74
		132		1		103		8	
·27	·04172		·0324		·05683		·0130		·73
		129		1		96		10	
·28	·04301		·0323		·05779		·0120		·72
		126		1		89		10	
·29	·04427		·0322		·05868		·0110		·71
		123		2		82		10	
·30	·04550		·0320		·05950		−·0100		·70
		120		2		75		11	
·31	·04670		·0318		·06025		·0089		·69
		117		4		68		12	
·32	·04787		·0314		·06093		·0077		·68
		114		4		61		11	
·33	·04901		·0310		·06154		·0066		·67
		111		5		54		12	
·34	·05012		·0305		·06208		·0054		·66
		107		5		48		13	
·35	·05119		·0300		·06256		−·0041		·65
		103		6		42		12	
·36	·05222		·0294		·06298		·0029		·64
		100		7		35		13	
·37	·05322		·0287		·06333		·0016		·63
		97		7		28		13	
·38	·05419		·0280		·06361		−·0003		·62
		92		8		23		14	
·39	·05511		·0272		·06384		+·0011		·61
		89		8		16		13	
·40	·05600		·0264		·06400		+·0024		·60
		85		9		10		13	
·41	·05685		·0255		·06410		·0037		·59
		80		9		5		14	
·42	·05765		·0246		·06415		·0051		·58
		77		10		2		13	
·43	·05842		·0236		·06413		·0064		·57
		72		10		7		14	
·44	·05914		·0226		·06406		·0078		·56
		67		11		12		13	
·45	·05981		·0215		·06394		·0091		·55
		63		11		18		14	
·46	·06044		·0204		·06376		·0105		·54
		59		12		24		13	
·47	·06103		·0192		·06352		·0118		·53
		54		11		29		13	
·48	·06157		·0181		·06323		·0131		·52
		49		12		34		13	
·49	·06206		·0169		·06289		·0144		·51
		44		13		39		12	
0·50	+·06250		+·0156		+·06250		+·0156		0·50
$c_3°$.	a_3.		b_3.		a_1.		b_1.		$c_1°$.

TABLE III.

$c_1°$.	$\mu_1 : \tau_2^2$.	$\mu_2 : \tau_2^2$.	$\mu_3 : \tau_2^2$.	$c_3°$.
0·20	+·06333	+·09667	−·03667	0·80
	117	49	216	
·21	·06216	·09716	·03451	·79
	119	47	214	
·22	·06097	·09763	·03237	·78
	121	46	213	
·23	·05976	·09809	·03024	·77
	123	44	211	
·24	·05853	·09853	·02813	·76
	124	43	209	
·25	·05729	·09896	−·02604	·75
	126	41	207	
·26	·05603	·09937	·02397	·74
	127	39	206	
·27	·05476	·09976	·02191	·73
	129	37	204	
·28	·05347	·10013	·01987	·72
	131	36	203	
·29	·05216	·10049	·01784	·71
	133	34	201	
·30	·05083	·10083	−·01583	·70
	134	33	199	
·31	·04949	·10116	·01384	·69
	136	31	197	
·32	·04813	·10147	·01187	·68
	137	29	196	
·33	·04676	·10176	·00991	·67
	139	27	194	
·34	·04537	·10203	·00797	·66
	141	26	193	
·35	·04396	·10229	−·00604	·65
	143	24	191	
·36	·04253	·10253	·00413	·64
	144	23	189	
·37	·04109	·10276	·00224	·63
	146	21	187	
·38	·03963	·10297	−·00037	·62
	147	19	186	
·39	·03816	·10316	+·00149	·61
	149	17	184	
·40	·03667	·10333	+·00333	·60
	151	16	183	
·41	·03516	·10349	·00516	·59
	153	14	181	
·42	·03363	·10363	·00697	·58
	154	13	179	
·43	·03209	·10376	·00876	·57
	156	11	177	
·44	·03053	·10387	·01053	·56
	157	9	176	
·45	·02896	·10396	·01229	·55
	159	7	174	
·46	·02737	·10403	·01403	·54
	161	6	173	
·47	·02576	·10409	·01576	·53
	163	4	171	
·48	·02413	·10413	·01747	·52
	164	3	169	
·49	·02249	·10416	·01916	·51
	166	1	167	
0·50	+·02083	+·10417	+·02083	0·50
$c_3°$.	$\mu_3 : \tau_2^2$.	$\mu_2 : \tau_2^2$.	$\mu_1 : \tau_2^2$.	$c_1°$.

TABLE IV.

$c_1°$.	$\lambda_0 : \tau_2^2$.	$\lambda_3 : \tau_2^2$.	$c_1°$.	$\lambda_0 : \tau_2^2$.	$\lambda_3 : \tau_2^2$.
0·20	+·0250	+·0750	0·50	0·0000	+·2500
	7	43		35	85
·21	·0257	·0793	·51	−·0035	·2585
	6	44		37	87
·22	·0263	·0837	·52	·0072	·2672
	6	44		41	91
·23	·0269	·0881	·53	0113	·2763
	5	45		44	94
·24	·0274	·0926	·54	·0157	·2857
	4	46		47	97
·25	·0278	·0972	·55	−·0204	·2954
	3	47		51	101
·26	·0281	·1019	·56	·0255	·3055
	3	47		54	104
·27	·0284	·1066	·57	·0309	·3159
	1	49		59	109
·28	·0285	·1115	·58	·0368	·3268
	1	49		64	114
·29	·0286	·1164	·59	·0432	·3382
	0	50		68	118
·30	·0286	·1214	·60	−·0500	·3500
	1	51		74	124
·31	·0285	·1265	·61	·0574	·3624
	3	53		79	129
·32	·0282	·1318	·62	·0653	·3753
	3	53		85	136
·33	·0279	·1371	·63	·0738	·3888
	4	54		92	142
·34	·0275	·1425	·64	·0830	·4030
	6	56		99	148
·35	·0269	·1481	·65	−·0929	·4178
	7	57		106	157
·36	·0262	·1538	·66	·1035	·4335
	8	58		115	165
·37	·0254	·1596	·67	·1150	·4500
	9	59		125	175
·38	·0245	·1655	·68	·1275	·4675
	11	61		135	185
·39	·0234	·1716	·69	·1410	·4860
	12	62		146	196
·40	·0222	·1778	·70	−·1556	·5056
	14	64		158	208
·41	·0208	·1842	·71	·1714	·5264
	15	65		172	222
·42	·0193	·1907	·72	·1886	·5486
	17	67		187	237
·43	·0176	·1974	·73	·2073	·5723
	19	69		204	254
·44	·0157	·2043	·74	·2277	·5977
	21	71		223	273
·45	·0136	·2114	·75	−·2500	·6250
	22	72		245	294
·46	·0114	·2186	·76	·2745	·6544
	25	75		268	319
·47	·0089	·2261	·77	·3013	·6863
	27	77		296	346
·48	·0062	·2338	·78	·3309	·7209
	30	80		327	377
·49	·0032	·2418	·79	·3636	·7586
	32	82		364	414
0·50	+·0000	+·2500	0·80	−·4000	+·8000

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The Width of Fraunhofer Lines. By John Q. Stewart, Assistant Professor of Astronomical Physics in Princeton University.

(*Communicated by Professor Henry Norris Russell.*)

1. In a recent important paper on "The Equilibrium of the Calcium Chromosphere," Professor Milne * has attributed the width of Fraunhofer lines (1) to the Doppler effect due to thermal molecular agitation, and (2) to the influence of random fields of neighbouring molecules on the frequency of response of the molecules active in producing the lines. He assumed that the effect (3) of "intrinsic width," due to lack of absolute sharpness in the quantum conditions, was negligible in comparison.† These three assumptions apparently were taken over from a paper by Tolman.‡ There is some reason to believe that all three are erroneous when applied to the rarefied gases of stellar atmospheres—that the broader Fraunhofer lines are much wider than can be accounted for either by Doppler effect, or by mutual molecular influences; and that their widths more probably can be ascribed to intrinsic lack of sharpness in the "tuning" of the active (scattering) molecules.

These matters have in part been discussed in a theoretical paper by the writer,§ in which an adaptation of a treatment due to Lorentz, based upon classical electron theory, leads to formulæ which indicate—(*a*) That in the solar atmosphere the Doppler effect is a negligible factor in accounting for the observed widths of lines corresponding to elements of even moderate atomic weight. (It may be appreciable, however, in the case of hydrogen.) (*b*) That Rayleigh scattering close to "resonance" is quite competent to account for the observed maximum widths (of the order of 1 Angstrom), and this on the assumption of very tiny quantities of scattering material.

2. *Doppler Widening.*—The argument regarding Doppler effect is essentially as follows:—Assuming a Maxwellian distribution of velocities among the molecules active in producing a line, the fraction of the molecules which possess speeds in the line of sight between u and $u+du$ is $(\epsilon^{-u^2/\alpha^2}/\sqrt{\pi})(du/\alpha)$, where ϵ is the base of natural logarithms, and α^2 is two-thirds of the mean square velocity of thermal agitation. The corresponding change, $\Delta\lambda$, in wave-length, λ, in emission or absorption, is $\lambda u/c$. For hydrogen at 5000° absolute, α is $9{\cdot}1\times10^5$ cm./sec., and for other substances α is inversely proportional to the square root of the molecular weight.

In order that $\Delta\lambda/\lambda$ may possess the value 4×10^{-5}, a sufficient number of molecules to produce appreciable selective opacity must be moving in the line of sight with speeds of the order at least $12{\cdot}0\times10^5$ cm./sec. In the case of hydrogen an appreciable fraction (about 1/140) of all the molecules possess speeds as great or greater than this. For calcium in the solar atmosphere, $\alpha=1{\cdot}44\times10^5$; and an utterly negligible fraction

* E. A. Milne, *Monthly Notices of R.A.S.*, **85**, 111.
† *Ibid.*, p. 118.
‡ R. C. Tolman, *Phys. Rev.*, **23**, 693, 1924.
§ J. Q. Stewart, *Astroph. Journ.*, **59**, 30, 1924.

of the calcium molecules have sufficiently high speeds to contribute, through Doppler effect, to the selective opacity for wave-lengths a few tenths of an Angstrom distant from the centre of a line.

3. *Mutual Molecular Influences.*—Two types of such are, (*a*) Stark effect; (*b*) effect of molecular collisions. Russell and Stewart * have developed a rough formula for estimating the broadening of H_β, in absorption, due to Stark effect of the electric fields of neighbouring molecules. The width thus caused in the solar spectrum is indicated as of the order 1 Angstrom. For elements other than hydrogen, however, the magnitude of the Stark effect appears to be very much less (although the data are scanty); consequently it may be tentatively concluded that this cause of line-widening is important, in the solar spectrum, only for hydrogen. Photographs show, indeed, that the hydrogen lines are unique in that they are the only wide lines in the solar spectrum which are not also conspicuously dark. This fact is compatible with the conclusion that peculiar factors operate to broaden these lines without increasing the effectiveness of the individual hydrogen atoms in producing opacity—spreading the opacity out thin, as it were, in the manner which Professor Milne assumed was characteristic of all broad lines.

Russell and Stewart considered that the effect of molecular collisions in broadening lines, in gases of the low density (10^{-5} atmospheres) of the solar reversing layer, also was negligible; this conclusion was indicated by the Lorentz treatment of impact damping. (This treatment was based on classical theory; but it is to be expected that quantum theory would lead to values of the same order of magnitude.)

While these considerations render it probable that mutual molecular influences are negligible in the greater part of the reversing layer, such causes of line-broadening may operate with appreciable effectiveness in the denser lower reversing layer, but the temperature there is so near that of the photosphere that the only consequence is a fuzzing of the edges of lines. On the other hand, at the extremely low densities of the chromosphere—to which Milne's study is applied—the molecules are so far apart that probably only the "intrinsic" line-width comes into question.

4. *Intrinsic Width.*—The well-known Rayleigh scattering formula has been satisfactorily confirmed by experiment as regards frequencies of radiation rather different from the "natural frequencies" which correspond to the Fraunhofer lines. Theoretically this formula should hold good for wave-lengths differing by as little as 0·1 Angstrom from the centre of a given line. Classical theory indicates also that, for a given density of active molecules, the index of refraction, μ, is much increased as regards such wave-lengths; the opacity due to Rayleigh scattering is proportional to $(\mu-1)^2$; † and the Rayleigh formula indicates that observable opacity should result with even a tiny quantity of active material per sq. cm. column in the line of sight. The Fraunhofer lines, according to this view, are the result of selective scattering rather than of true absorption.

* H. N. Russell and J. Q. Stewart, *Astroph. Journ.*, **59**, 204, 1924.
† E. A. Milne, *Monthly Notices of R A.S.*, **85**, 118.

Another way of putting the matter is that a given molecule, regarded as a resonator of the type suggested by classical electron theory, responds somewhat to radiation of any frequency; but the response sharply increases as the impressed frequency is brought closer to the natural frequency (in the region where anomalous dispersion is observed); and consequently the number of molecules required to produce appreciable opacity is greatly reduced. Evidently this lack of absolute sharpness of tuning is equivalent, in the terminology of quantum theory, to "intrinsic width."

5. *Milne's Equation for the Life of an Excited State.*—The objection may be raised to this conclusion regarding the importance of "intrinsic width" that it is based on wave-theory of light and on classical electron theory, and thus may be in error, owing to the neglect of quantum considerations. The particular quantum theory of scattering employed in Milne's paper (adapted from Tolman's treatment) leads, however, to essentially the classical results—as expressed in equation (24),* to which the preceding quantum equations lead; and on which much of the theory of chromospheric conditions is based—

$$F_\nu = \frac{8\nu^2\tau mg}{c}.$$

This equation does not contain h; and Milne suggests that it can be derived classically. In fact it is equivalent to the equation deduced by Page † from the classical theory of the simple linear electronic oscillator,

$$P = \tfrac{1}{2}\pi a F_\nu,$$

where P is the average total "radiation pressure" (force) on a molecule due to resonant scattering, close to frequency ν, of "white" radiation of intensity $F_\nu d\nu$; and a is the radius of the electron calculated from the usual formula (or $1{\cdot}88\times10^{-13}$ cm.). Well-known electromagnetic theory shows that

$$a = \frac{c}{4\pi^2\nu^2\tau},$$

where τ is the time-interval (of the order 10^{-8} sec. for visible radiation) which defines the electromagnetic damping of the oscillator. Substituting for a above, and equating P to mg, the weight of the molecule of mass m just supported against solar gravity by radiation pressure, we get (except for a factor of π) Milne's equation. Thus this equation is immediately implied in the classical formulæ employed by the writer.

The experiments of Füchtbauer and his collaborators (dealt with in Tolman's paper ‡) on the selective opacity associated with the formation of (so-called) absorption lines by vapours of the alkali metals, indicate that the predictions of classical theory, dealing with resonance scattering, are at least not greatly in error as regards order of magnitude.

6. *Line-Width and Radiation Pressure.*—Professor Milne's conclusion that the radiation-pressure in the solar chromosphere supports against

* E. A. Milne, *Monthly Notices of R.A.S.*, **85**, 122.
† L. Page, *Astroph. Journ.*, **52**, 70, 1920.
‡ R. C. Tolman, *Phys. Rev*, **23**, 693, 1924.

solar gravity enough ionised calcium to account for the observed wide H and K lines is, as he points out, in disagreement with that of the writer, namely, that even the tiny amount of material necessary (on the hypothesis that the observed line-width is due to Rayleigh scattering near resonance) to account for widths of Fraunhofer lines is much greater than can be supported by radiation-pressure against solar gravity. Milne's statement is not, however, based upon the mathematics of his paper; it is implied in his initial assumption that the widths of the calcium lines are accounted for by the Doppler effect and mutual molecular influences—factors not dealt with in his later argument.

Thus, the width of a strong line of ionised calcium is taken (from observational data) as 1 Angstrom *; then, from the known intensity of solar radiation of the appropriate frequency, the mass of calcium supported, when a given small fraction of the radiation emerges from the solar atmosphere, is immediately calculable from the elementary formula for radiation-pressure (regardless of the value of τ). There is nothing in the paper, however, to show that the small mass thus supported is sufficient to produce the broad calcium lines observed, and the considerations mentioned above indicate that it is not nearly sufficient.

Thus the proposition advanced by the writer is not affected by Milne's study, namely, that practically all of the Fraunhofer absorption is produced in the reversing layer, where radiation-pressure is small as compared with gas-pressure.

7. *Electrostatic Field.*—One reservation, however, is to be made regarding this statement; the electrostatic field in the solar atmosphere, which Pannekoek described,† may be capable of supporting ions at high levels which otherwise would come to equilibrium lower down. Incidentally, Pannekoek's treatment applied only to a mixture of free electrons and ions of a single type; where several types of ions are present, an obvious modification of his treatment indicates that the electrostatic levitation on ions less heavy than the average in a given region, or more highly charged, may be actually stronger than solar gravity. The analysis follows :—

Suppose that in a layer of unit area and infinitesimal thickness, dz (where z is counted as positive in the upward direction), forming part of a unit column in the solar atmosphere, there are dn atoms of element, ζ, each possessing mass m, and average positive charge (with respect to time, or individual atom chosen), ie, where e is the electronic charge. (For example, i is 2 if all the atoms, ζ, are doubly ionised.) The downward pull of gravity per atom is mg, where g is the acceleration of gravity in the given region. Suppose a vertical electrostatic field is present, of intensity F, counted as positive if directed upward; then the decrease in the partial pressure of the atoms of the given type, between the bottom and the top of the infinitesimal layer, may be written by

$$dp = -mgdn + ieFdn \quad . \quad . \quad . \quad . \quad (1)$$

* E. A. Milne, *Monthly Notices of R.A.S.*, **85**, 128.
† A. Pannekoek, *B.A.N.*, No. 19, 1922.

(This equation is based on the assumption that the equilibrium of each type of atoms can be described without reference to that of the other types provided only that F is given.)

Similarly, for the free negative electrons, of number dn', and mass m',

$$dp' = -m'gdn' - e\mathrm{F}dn' \quad . \quad . \quad . \quad (2)$$

The number of negative electrons in the layer is very nearly equal to its average gross positive charge in electronic units (since gravity, as Pannekoek pointed out, is unable to effect much separation between the charges); that is,

$$dn' = \Sigma i dn, \quad . \quad . \quad . \quad . \quad (3)$$

very nearly, where the summation is taken over all the types of atoms, ζ.

Further, referring to partial pressures,

$$p' = \Sigma ip\,;$$

so that

$$dp' = \Sigma i dp + \Sigma p di \quad . \quad . \quad . \quad . \quad (4)$$

Combining (1), (2), and (4),

$$g\Sigma m i dn - e\mathrm{F}\Sigma i^2 dn - \Sigma p di = m'gdn' + e\mathrm{F}dn' \quad . \quad . \quad (5)$$

The gas law,

$$pdz = \mathrm{RT}dn$$

(where T is the absolute temperature, and R the gas constant per molecule), leads to

$$\Sigma p di = \mathrm{RT}\sum\left(\frac{1}{i}\frac{di}{dz}\right) i dn = \mathrm{RT}\Sigma \mathrm{L} i dn \quad . \quad . \quad (6)$$

Write

$$\left.\begin{aligned} \Sigma m i dn &= \bar{m}\Sigma i dn \\ \Sigma i^2 dn &= \bar{i}\Sigma i dn \\ \Sigma \mathrm{L} i dn &= \bar{\mathrm{L}}\Sigma i dn \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

Then, combining (5), (6), (7), and employing (3),

$$e\mathrm{F} = \frac{(\bar{m} - m')g - \mathrm{RT}\bar{\mathrm{L}}}{1 + \bar{i}} \quad . \quad . \quad . \quad (8)$$

The "effective weight" of an atom of type ζ is,

$$\mathrm{W} = \frac{g}{1+\bar{i}}(m + m'i + m\bar{i} - \bar{m}i + i\mathrm{RTL}/g)\,; \quad . \quad . \quad (9)$$

and of a free electron,

$$\mathrm{W}' = \frac{g}{1+\bar{i}}(\bar{m} + m'\bar{i} - \mathrm{RT}\bar{\mathrm{L}}/g) \quad . \quad . \quad . \quad (10)$$

In these equations m' may be neglected in comparison with m and $\bar{m}$.

The total weight of the infinitesimal layer of ionised gas is given by

$$W'dn'+\Sigma W dn=[m'dn'+\Sigma m dn]g;$$

which, of course, is independent of the condition of ionisation.

Full discussion of (9) involves a knowledge of $\bar{L}$, which in turn requires a fairly complete understanding of the physical conditions of the solar atmosphere. If the ionisation is greater in the upper regions, $\bar{L}$ is positive; for example, if the average ionisation increases from 0 to 1 in 100 km., $\frac{di}{dz}$ is 10^{-7}. For T = 5000° absolute, RT is 7×10^{-13}; at the sun's surface g is $2{\cdot}67\times10^{4}$. Consequently the value of the term $i\mathrm{RT}\bar{L}/g$ may be of the order of 10^{-24}. This is of the order of magnitude of atomic masses.

Neglecting, however, the term in $\bar{L}$, (9) indicates, for atoms of a given element in a given region of the star, that if m is much smaller than $\bar{m}$, or i much greater than $\bar{i}$, the effective weight is negative, and the atoms will be moved upward.

The whole matter requires more detailed discussion; but it is evident that electrostatic forces may be an even more important factor in the equilibrium of stellar atmospheres than Pannekoek supposed. Rosseland * and Milne † have elaborated the theory of electrostatic forces in stellar atmospheres, but neither of these papers has brought out explicitly the points emphasised here.

8. *The Three Factors concerned in the Formation of Fraunhofer Lines.*—A theory of the formation of Fraunhofer lines must take into account the following three factors: (1) the selective opacity of a given region in a stellar atmosphere; (2) the selective emissivity, as effected by the negative temperature gradient outward from the photosphere; (3) fluorescence, or the effect on emissivity of the transformation in the frequency of radiation by sub-atomic mechanisms. The simultaneous action of the first two factors was recognised as essential by Kirchoff. The third is of less general importance, but may be invoked to explain lines which are either brighter or darker than the simple thermodynamic theory indicates. Neglecting fluorescence, the intensity of radiation emergent from a stellar atmosphere at the centre of any Fraunhofer line should not be less than that of black-body radiation of the given frequency and corresponding to the temperature of the bounding region of the atmosphere, nor greater than the intensity of photospheric radiation of the given frequency.

9. "*Effective Depth.*"—In this connection, a useful idea is that of the "effective depth" in a stellar atmosphere corresponding to emergent radiation of a given frequency. This may be defined as the depth such that the integrated opacity in the line of sight for the given narrow frequency band corresponds to (say) a 50 per cent. reduction in the intensity of the direct beam. Then the elementary thermodynamic theory, which neglects fluorescence, indicates that the intensity of the

* S. Rosseland, *M.N.*, **84**, 720.
† E. A. Milne, *Proc. Cambridge Phil. Soc.*, **22**, 493, 1925.

emergent radiation should be about that of the corresponding black-body radiation at the temperature of the region at the " effective depth." The effective depth is comparatively small at the centre of any line, but increases at the edges to that of the photosphere. The width of the line is a matter of the nature of the variation over the small frequency-range of the selective opacity of the stellar atmosphere. The blackness, however, is only secondarily, through the " effective depth," a function of selective opacity ; primarily the blackness is a contrast effect dependent upon the temperature difference between the photosphere and the region at the effective depth.

The width of a line increases steadily, although not in direct proportion, with the number of molecules active in its production. The blackness of a line, however, on this simple theory, reaches saturation when the region where the line is produced has risen to the outer and cooler portions of the stellar atmosphere. Thus, owing to the temperature gradient, a given number of active molecules are more efficient in producing a dark line when they lie in a region comparatively high above the photosphere.

10. *General* versus *Selective Opacity.*—The effective depth of the photosphere depends upon the general opacity of the gas, which does not vary sharply with the frequency, and which increases with the density and temperature of the gas, since it depends upon mutual molecular influences. The " effective depth " for the centre of a Fraunhofer line is determined by the corresponding selective opacity, which is, per reacting molecule, to a first approximation unaffected by density and temperature. The writer has given a brief qualitative treatment of the effect on line-intensities of this relationship between general and selective opacity.*

11. *Classical* versus *Quantum Theory of Opacity.*—The problem of opacity, general as well as selective, falls within a subdivision of the science of physics which at present is in an unsatisfactory condition. The whole matter demands careful treatment, and, in particular, the accumulation of experimental data. The most reliable results probably are to be obtained by employing classical theory jointly with quantum theory in correlating the meagre data at present available.

In conclusion, it may be pointed out that, in so far as exception is taken in the present paper to statements made in the paper of Professor Milne, the difference is not one relating to matters of fact or of analysis, but solely to certain assumptions which, in Professor Milne's paper, are clearly stated as such ; and the contention is merely that, in the light of available evidence, better assumptions might have been made.

* J. Q. Stewart, *Popular Astronomy*, **33**, 303, 1925.

The Width of Fraunhofer Lines: a Reply to Professor Stewart.

By Professor E. A. Milne.

Professor Stewart's courteous criticisms of my paper on "The Equilibrium of the Calcium Chromosphere" are concerned principally with the part played by line-width in my investigation. I am indebted to him for pointing out that the fundamental formula for the residual intensity in a line (when the atoms absorbing the line are supported by radiation-pressure) can also be derived, as I surmised, from classical theory, and with much of his discussion in §§ 8–11 I am in full agreement. The same ideas were in fact employed in my paper, in so far as it was concerned with the part played by the reversing layer in producing absorption lines. But I am unable to agree with him that my conclusions as to the equilibrium of the chromosphere depend on any particular assumptions as to line-width. The point of my method is that it makes a sharp distinction between line-width and residual intensity, and it shows that the formation, degree of permanence, and law of density-distribution of the chromosphere are independent of any knowledge concerning the breadths of spectral lines. The only place where I employed a value of a line-width $\Delta\nu$ was when it was a question of estimating the actual mass of atoms in a column of unit cross-section; and I then took a value of $\Delta\nu$ corresponding to $\Delta\lambda=1$ A.U., not from observation, as Professor Stewart supposes, but merely by way of illustration, to have definite numbers before me (p. 126). These gave the orders of magnitude of the density, total number of atoms per cm.^{2}, etc., but I pointed out that if any other value of $\Delta\nu$ were adopted these numbers would simply have to be multiplied proportionally. The residual intensity, on which alone the conclusions of the paper depended, would be unaltered. The *actual* densities in the upper chromosphere probably correspond more nearly to $\Delta\lambda=0{\cdot}1$ A.U. or less.* It seems worth while examining the point in detail.

1. At the beginning of the paper, I attributed the "finite breadth of an observed spectral line" to (*a*) intrinsic breadth depending on the structure of the atom, (*b*) the effect of neighbouring atoms and of the Doppler effect. It was a question of making certain fundamental definitions, and the context shows that the circumstances which were being discussed were those in the ideal case of a beam of light incident on a thin layer of gas—the ideal experiment by which an atomic absorption coefficient could be measured in the laboratory. It is well known that in resonance radiation—for example, the radiation emitted by mercury vapour or sodium vapour when acted on by $\lambda2537$ or D-radiation respectively—the lines are extremely narrow. If we could slice off a piece of the upper chromosphere and examine the radiation emitted by the atoms, we should similarly find the H and K radiation to be confined within very narrow lines, far narrower than the Fraunhofer H

* In my earlier paper, "An Astrophysical Determination of the Average Life of an Excited Calcium Atom," *M.N.*, **84**, 358, 1924, I adopted $\Delta\lambda=0{\cdot}1$ A.U. in estimating the mean mass absorption coefficient of Ca^+ atoms for H and K radiation.

and K lines. This is implied by everything we know in the laboratory about resonance radiation, and the high-level Ca^+ atoms must necessarily be resonating, *i.e.* absorbing and re-emitting, under the influence of radiation from below.

It is these narrow lines with which alone my investigation is concerned, and effects (*a*) and (*b*) above exhaust the possibilities. Professor Stewart suggests that effect (*a*) may give rise to the order of magnitude of the observed widths of Fraunhofer lines. My reasons for believing this suggestion to be erroneous are given in § 3 below.

Whatever the width, and from whatever cause the width originates, the Einstein probability coefficients take account of it. In my analysis it is shown that the integral of the atomic absorption coefficient $a(\nu)$ through the line is given by

$$\int a(\nu)d\nu = \frac{B_{1\to 2}h\nu_0}{4\pi} = \frac{h\nu_0}{4\pi\tau}\cdot\frac{c^2}{2h{\nu_0}^3} = \frac{c^2}{8\pi\tau{\nu_0}^2} \quad . \quad . \quad (1)$$

In this equation, which is simply a consequence of the existence of a state of thermodynamic equilibrium in a closed chamber containing atoms and radiation, all sources of broadening under the circumstances holding in the chamber are necessarily taken into account.

It will thus be seen that when as an approximation we put

$$\int a(\nu)d\nu = a\Delta\nu,$$

the $\Delta\nu$ of this formula has nothing to do with the actual width of the corresponding Fraunhofer line in the solar spectrum. It refers only to a narrow strip embedded in the middle of the observed broad Fraunhofer line, in which strip alone the atoms at high levels can be absorbing and emitting. This was stated explicitly on p. 135 of my paper.

I am well aware of the difficulties of framing a theory to account for the observed width of Fraunhofer lines in general, still more of the H and K lines in the solar spectrum, and of the Balmer lines in A-type stars. The Doppler effect, for example, is well known to be inadequate. But whatever the nature of the phenomenon of broadening, it must have its seat in the reversing layer, where the conditions are relatively complicated. In considering the high-level chromosphere we are not concerned with the formation of a spectrum by the reversing layer. We simply accept this spectrum—a continuous spectrum crossed with lines—as being the radiation incident on the high-level atoms from below.

2. Professor Stewart appears to have misunderstood me (§ 6 of his paper), when he says that I have concluded "that the radiation-pressure in the solar chromosphere supports against solar gravity enough ionised calcium to account for the observed wide H and K lines" (his words). This was not my conclusion. What I showed was that "mass sufficient to give an intensity-ratio of $\frac{1}{9}$ can be supported by radiation-pressure if $\tau = 0{\cdot}6\times10^{-8}$ sec.," and that the difference from Professor Stewart's result was that my calculation showed "no 500-fold discrepancy between the mass required to produce the residual intensities in the centres

of the H and K absorption lines and the mass that can be supported by radiation-pressure." My claim was that the theory accounted for the observed residual intensities at the centres. Naturally no theory of the chromosphere only could account for the observed widths; as already remarked, the line-width relevant to the high-level atoms must be very small, and the observed widths of the Fraunhofer lines must be due to their mode of formation in the reversing layer.

The theory showed that Ca^+ atoms must necessarily be expelled from the sun if the residual intensity at the centre of the lines exceeds $\frac{1}{9}$ of the continuous spectrum.* Consequently in a steady state the residual intensity cannot exceed this value. Since the residual intensity calculated from the boundary temperature (*cf.* Stewart, § 8) was shown in my paper (p. 123) to be 0·35 of the neighbouring continuous spectrum, the pressure of radiation must bring into existence a screen of atoms at least thick enough to reduce the intensity-ratio to $\frac{1}{9}$. Such extra reduction in intensity will only affect the centre of the line, since at high levels the atoms are only absorbent through a narrow spectral region at the centre. We thus have the contour of the absorption produced by the gases in the reversing layer, and superimposed on this the additional darkening just mentioned, due to atoms fully supported by radiation-pressure. This latter factor governing the residual intensity in a Fraunhofer line is not amongst those enumerated by Stewart in §§ 8, 9 of his paper, but it is difficult to see how it can avoid occurring. Stewart's discussion here assumes a state of thermodynamic equilibrium at each level, which does not hold for the upper chromosphere. (Cf. *Nature*, **113**, 534, 1924 April 12.)

When we know τ for an atom, we know how fast it can absorb and re-emit quanta when exposed to a given intensity of radiation. Hence we know how fast it is absorbing outward momentum, and it is simply a question of how large this is compared with the rate of acquisition of downward momentum under gravity. My conclusion is that if the residual intensity were that corresponding to the probable temperature of the reversing layer, the rate of absorption of outward momentum would be the larger. Hence the existence of a chromosphere—a mass of atoms supported by radiation-pressure.

Stewart's conclusion † was that to produce a certain width for the D-lines of sodium, 500 times as many atoms were necessary as could be supported by the radiation-pressure arising from the absorbed radiation. He does not inquire whether the topmost atom of his column may not be acted on by a radiation-pressure greater than gravity. In my procedure, no consideration of the number of atoms required is relevant. The residual intensity must be cut down to the necessary value, through whatever small range $\Delta\nu$ the high-level atoms are sensitive. If we double $\Delta\nu$, twice as many atoms will be supported to occupy the double range, but the residual intensity will be unaltered.

* This way of putting it assumes $\tau = 0{\cdot}6 \times 10^{-8}$ sec. Actually the observed ratio $\frac{1}{5}$ was used to deduce this value of τ, but a value of the order of 10^{-8} sec. could well have been assumed from Wien's laboratory investigations on other atoms.

† *Astrophys. Journ.*, **59**, 35, 1924.

3. As regards the origin, as distinguished from the nature, of our disagreement, Professor Stewart uses a formula *

$$N = 1{\cdot}5 \times 10^{24} k \Delta^2/\lambda^2 \quad . \quad . \quad . \quad . \quad (3)$$

giving the number of molecules N per unit area necessary to produce a line of wave-length λ and observable width Δ, k being such that the intensity-ratio in the line has the value† $1/(1+k)$. Relation (3) is deduced from the classical electromagnetic theory. It represents the broadening due to the response of a resonator to frequencies different from the one to which it is tuned. Given some intrinsic source of broadening of this kind, a line as broad as we please can be produced by sufficiently increasing N. Professor Stewart in §1 of the preceding note claims that formula (3) is "quite competent to account for the observed maximum widths (of the order of 1 A.U.), and this on the assumption of very tiny quantities of scattering material." But it appears to me that his calculation simply shows that *if* this type of broadening is to account for the observed widths, then 500 times as many atoms are required as can be supported by radiation-pressure. His calculation might equally well be taken to show that if the atoms producing the line are in fact supported by radiation-pressure, then the observed widths are about $\sqrt{500}$ times as big as is given by (3). I think the following calculation will show that the latter is the true explanation, though the real discrepancy is greater still.

Professor Stewart deduces (3) from a formula ‡

$$a_\nu = \frac{a_0}{1 + A(\nu - \nu_0)^2} \quad . \quad . \quad . \quad . \quad (4)$$

where a_ν is the atomic absorption coefficient for radiation of frequency ν, and a_0 and A are constants having the values

$$a_0 = \frac{c^2}{2\pi {\nu_0}^2}, \quad A = \frac{c^2}{\pi^2 a^2 {\nu_0}^4},$$

where a is the radius of the electron, $\frac{2}{3}e^2/mc^2 = 1{\cdot}88 \times 10^{-13}$. The constant A has a simple physical meaning. If $\nu_0 \pm \delta\nu$ is the frequency for which a_ν becomes half its value for $\nu - \nu_0$, we may describe $\delta\nu$ as the "intrinsic half-width" of the line. (It has, of course, nothing to do with the "observed width.") Then clearly

$$A(\delta\nu)^2 = 1$$

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* *Astrophys. Journ.*, **59**, 33, 1924. Also Russell and Stewart, *ibid.*, **59**, 202, 1924.

† Stewart uses $1/(1+\frac{1}{2}k)$. But it can be shown that Schuster's factor $\frac{1}{2}$ occurs through his using in effect a mean absorption coefficient equal to twice the true absorption coefficient.

‡ *Loc. cit.*, formula (2). I have, however, omitted a factor 3, in order to preserve agreement with the formula $\tau = c/4\pi^2 a {\nu_0}^2$ which he has given in § 5 of the preceding note. To simplify the work, I have here specifically omitted the Doppler effect. That this makes no ultimate difference will appear below when I show that, however the observed widths of the lines arise, the atoms necessary can be supported by radiation-pressure.

or $$\delta\nu = A^{-\frac{1}{2}}.$$

For $\lambda_0 = 6000$ A.U., we find that the $\delta\lambda$ corresponding* to $\delta\nu$ is $5{\cdot}9 \times 10^{-5}$ A.U. Also $a_0 = 5{\cdot}73 \times 10^{-10}$.

Formula (1) above must be satisfied whatever the source of broadening. Thus we ought to have

$$\frac{e^2}{8\pi\tau\nu_0^2} = \int_0^\infty a_\nu \delta\nu = \frac{\pi a_0}{\sqrt{A}} = \pi a_0 \delta\nu \quad . \quad . \quad . \quad (5)$$

and substituting for a_0 and A this gives

$$\tau = e/4\pi^2\nu_0^2 a,$$

in agreement with a formula given by Stewart in the preceding note. This is mentioned to show the consistency of (4) with our (1). For $\lambda = 6000$ A.U., we find the very reasonable value $\tau = 1{\cdot}62 \times 10^{-8}$ sec.

Now suppose there are N atoms per cm.^{2}. Then the optical thickness σ_ν for ν-radiation is given by

$$\sigma_\nu = N a_\nu,$$

and the residual intensity F_ν is given in terms of the incident intensity I_ν by the formula

$$I_\nu = F_\nu(1 + \sigma_\nu).$$

Thus

$$\frac{F_\nu}{I_\nu} = \frac{1}{1 + N a_\nu} = \frac{1 + A(\nu - \nu_0)^2}{1 + N a_0 + A(\nu - \nu_0)^2} \quad . \quad . \quad . \quad (6)$$

It is clear both physically and from this formula that we can make F_ν/I_ν as small as we please, for $\nu - \nu_0$ as large as we please, by taking N sufficiently large. Accordingly, defining the "observable half-width" $\Delta\nu$ as such that at $\nu_0 \pm \Delta\nu$, F_ν/I_ν has the value $\frac{1}{2}$, Professor Stewart finds that the value of N necessary to give this $\Delta\nu$ is given by

$$\tfrac{1}{2} = \frac{1 + A(\Delta\nu)^2}{1 + N a_0 + A(\Delta\nu)^2}$$

or

$$N a_0 = 1 + A(\Delta\nu)^2 = 1 + (\Delta\nu/\delta\nu)^2 \quad . \quad . \quad . \quad (7)$$

This is in effect Stewart's formula (3) above. Taking $\lambda = 6000$ A.U., $\Delta\lambda = 0{\cdot}3$ A.U. following Stewart, I find from (7) $N = 4{\cdot}50 \times 10^{16}$, agreeing well enough with Stewart's 3×10^{16}.

But if the theory is right, the whole contour of the absorption line is known once N is known. In particular, on putting $\nu = \nu_0$ in (6) we should have the intensity-ratio at the centre of the line, in the form

$$\left(\frac{F_\nu}{I_\nu}\right)_{\nu=\nu_0} = \frac{1}{1 + N a_0} = \frac{1}{2 + (\Delta\nu/\delta\nu)^2} \quad . \quad . \quad . \quad (8)$$

On inserting the values, we find this is $3{\cdot}9 \times 10^{-8}$, at least a million times smaller than the usually given order of magnitude of $(F_\nu/I_\nu)_{\nu=\nu_0}$ for

* We have the simple formula $\delta\lambda = c\delta\nu/\nu_0^2 = \pi a$.

Fraunhofer lines. In order to make the classical intrinsic width able to explain the observed widths, Professor Stewart has crowded so many atoms into the line of sight that the residual intensity in the centre of the line would be to all intents and purposes zero. If this were the case, it would be impossible to obtain spectroheliograms.

The correct inference from this discrepancy is that the classical value of $\delta\nu$ is too small to account for the observed $\Delta\nu$. This would have been Professor Stewart's conclusion if he had investigated the intensity-ratio at the centre of the line as well as at the edges. For he in no way showed that his theory *accounted* for the observed width; he adduced no reason why his value of N should be accepted save that it was "very tiny." Actually it is not acceptable because it gives the wrong value for the central intensity. If we take the central intensity-ratio $(F_\nu/I_\nu)_{\nu=\nu_0}$ as $\frac{1}{10}$, from observation, then (8) gives $(\Delta\lambda/\delta\lambda)^2=8$, and thus the predicted "observable half-width" is only $\Delta\lambda=1{\cdot}67\times10^{-4}$ A.U. This is 1/2000 of that actually observed.

Professor Stewart may object that from its mode of derivation (4) is definitely not valid at $\nu=\nu_0$. But a conclusion of the same character follows if we use an experimental value of 10^9 for the *mean* absorption coefficient through the line. The optical mass of the $4{\cdot}5\times10^{16}$ atoms would be

$$4{\cdot}5\times10^{16}\times23\times1{\cdot}64\times10^{-24}\times10^9=1700,$$

and so the *mean* residual intensity would be 1/1701 of the photospheric, as against the observed central intensity of $\frac{1}{10}$.

It is possible, however, to go further and actually show that whatever may be the unknown cause which produces lines of the observed breadth and central intensity, the mass necessary is of the order that can be supported by radiation-pressure. Let us now adopt as an empirical formula for the atomic absorption coefficient

$$a_\nu=\frac{a'_0}{1+A'(\nu-\nu_0)^2} \quad . \quad . \quad . \quad . \quad (9)$$

where a'_0 and A′ are constants to be determined to fit the observations. The one thing we know about a'_0 and A′ is that they must be consistent with (1). That is to say, whatever the physical causes which at one time or another broaden the line, $\int a_\nu d\nu$ always has the same value; differing physical circumstances simply spread out the range in which the atoms can absorb, leaving their intrinsic absorbing powers unaltered. Hence, repeating the calculation of equation (5), we have

$$\frac{a'_0}{\sqrt{A'}}=\frac{a_0}{\sqrt{A}} \quad . \quad . \quad . \quad . \quad (10)$$

Now assume that in the centre of the line the intensity-ratio is $\frac{1}{10}$, and that where $\Delta\lambda=0{\cdot}3$ A.U. it has altered to $\frac{1}{2}$. We have then

$$\frac{1}{10}=\frac{1}{1+N'a'_0}, \quad \frac{1}{2}=\frac{1+A'(\Delta\nu)^2}{1+N'a'_0+A'(\Delta\nu)^2} \quad . \quad (11), (12)$$

Equations (10), (11), (12) determine the three unknowns, N', a'_0, A'. They reduce to

$$a_0'\delta'\lambda = a_0\delta\lambda = 3{\cdot}38\times10^{-22},$$
$$N'a_0' = 9,$$
$$(\Delta\lambda/\delta'\lambda)^2 = 8,$$

the solution of which is

$$N' = 2{\cdot}825\times10^{13},$$
$$a_0' = 3{\cdot}19\times10^{-13},$$
$$\delta'\lambda = 0{\cdot}106 \text{ A.U.}$$

We see that an observed absorption-line contour implies a definite number of atoms N' and a definite intrinsic half-width $\delta'\lambda$. Our value for N', the number of atoms, is about a thousand times smaller than that found by Stewart. The weight of $2{\cdot}825\times10^{13}$ sodium atoms in the sun is

$$2{\cdot}825\times10^{13}\times23\times1{\cdot}64\times10^{-24}\times2{\cdot}74\times10^{4} = 2{\cdot}82\times10^{-5} \text{ dyne.}$$

According to Stewart, the available radiation pressure from the radiation obstructed in the production of a line of the width we have taken is $5{\cdot}8\times10^{-5}$ dyne. So far from there being 500 times as much sodium as could be supported by radiation-pressure, we find only half as much.

Of course we have only been concerned with orders of magnitude. The value of $\int a\,dv$ given by (5) is found to be $8{\cdot}85\times10^{-3}$, whereas Fuchtbauer and Hoffman's experimental value,* as quoted by Tolman,† is $0{\cdot}65\times10^{-3}$ for a D-line. This is 13·6 times smaller, and leads to a value of N 13·6 times bigger. This would give a weight of sodium about 7 times bigger than the radiation-pressure, but it appears to me that Stewart has omitted a factor π in his computation of the radiation-pressure. But these are minor considerations. It is true that the number of atoms required is tiny, but it is 1000 times tinier than he supposed. The main point is that the calculated mass of sodium necessary to produce an observed line, not only 0·6 A.U. wide but also with an intensity-ratio of $\frac{1}{10}$ at the centre, is of the order that can be supported by radiation-pressure.‡ Professor Stewart's contrary conclusion arose from his not taking into account the second requirement. A given intrinsic width $\delta\nu$ can be made responsible for as big an observed width $\Delta\nu$ as we please by taking N large enough, but only at the cost of cutting down the central intensity far below what is observed. It is therefore misleading to speak of line-width as arising from an "increase in the number of atoms in the line of sight," § without any other qualification.

It should be noticed that our empirical treatment based on a formula of the type (9) is independent of the specification of particular causes of broadening, such as the Doppler effect. Any source of approximately

* *Ann. der Phys.*, **43**, 96, 1914. † *Phys. Rev.*, 1924 June, p. 702.

‡ As for Ca^+ atoms, there must be a relation between τ and the central intensity-ratio.

§ Stewart, *loc. cit.*, p. 30; Russell and Stewart, *loc. cit.*, p. 201.

symmetrical broadening will give rise to a formula of the type (9). For example, the particular formulæ used by Stewart to take account of the Doppler effect are of the form (9) when $A'(\nu-\nu_0)^2$ is small and when it is large, respectively.

Our argument in this § 3 has proceeded as though the parameters fixing the atomic absorption coefficient were the same at all levels. Actually, however, the absorbing power will be more spread out at the lower levels ; α_0 and A must be functions of level, decreasing with increasing depth. The values we have found are to be considered as averages. Our conclusion is that in the reversing layer there is some physical circumstance causing the atoms to emit and absorb as though they had an intrinsic half-width $\delta'\lambda$ of the order of 0·1 A.U. Presumably the excess of this over the classical intrinsic width plus Doppler broadening is due in some way to the presence of other atoms and electrons, though no known physical effect seems adequate.

4. I have followed Stewart's method in his paper in the *Astrophysical Journal*, in applying Schuster's formula $F_\nu=I_\nu(1+\sigma_\nu)$. We thereby assume a local radiative equilibrium in the radiation of each frequency ν —what Professor Stewart would call scattering. The ideal circumstances to which the Schuster analysis refers are those of a layer of atoms in front of a plane which is radiating with a given intensity-distribution I_ν. It is important to emphasise that this theory of the production of a Fraunhofer line is essentially different from the theory dwelt on by Professor Stewart in §§ 8, 9, 10 of the preceding paper. In the latter, we are simply looking into stratified material of infinite thickness through which radiation of various frequencies is pouring. It is unequally obstructed in different frequencies, hence absorption lines. There is no internal radiating surface—the intensity emergent in any frequency is simply that corresponding to the average depth to which we can see in that frequency. The emergent intensity in any frequency is no longer given by a Schuster formula, for I_ν has no meaning. There is no longer an equilibrium for each separate frequency. But it is easy to derive a formula which represents the facts, as follows :—

Let α be the general coefficient of opacity, α_ν the selective coefficient in the neighbourhood of a line. Thus if the line is $\nu=\nu_0$, $(\alpha_\nu/\alpha)_{\nu=\nu_0}$ is very large, but $\alpha_\nu\to\alpha$ as ν departs from ν_0. Let B_ν be the intensity of black-body ν-radiation for the temperature at the optical depth σ. The temperature T is a function of σ, and hence B_ν may be considered to be a function of σ. The optical depth σ is supposed reckoned in terms of α. Then the usual approximate equation of transfer * for the outward intensity I_ν,

$$\frac{1}{2}\frac{dI_\nu}{d\sigma}=\frac{\alpha_\nu}{\alpha}(I_\nu-B_\nu),$$

gives for the emergent intensity F_ν in the neighbourhood of the line the expression

$$F_\nu=\int_0^\infty B_\nu e^{-2(\alpha_\nu/\alpha)\sigma}2(\alpha_\nu/\alpha)d\sigma,$$

* *M.N.*, **81**, 377, 1921.

whilst the adjacent continuous background is given by

$$G_\nu = \int_0^\infty B_\nu e^{-2\sigma} 2 d\sigma.$$

To obtain a definite expression, let us now suppose that as an approximation B_ν may be taken as a linear function of σ, say

$$B_\nu = (B_\nu)_0(1+b_\nu\sigma),$$

where $(B_\nu)_0$ is the intensity of black-body ν-radiation for the boundary temperature.*

We then find

$$F_\nu = (B_\nu)_0\left(1+\tfrac{1}{2}b_\nu\frac{\alpha}{\alpha_\nu}\right)$$

$$G_\nu = (B_\nu)_0(1+\tfrac{1}{2}b_\nu).$$

From these two formulæ, the various known results stated by Professor Stewart in §§ 8, 9, 10 follow immediately. Thus if α_ν is large compared with α, as it is in the centre of the line, F_ν is practically equal to $(B_\nu)_0$, the black-body intensity corresponding to the boundary temperature. Again, the effective depth for the photospheric radiation is given by

$$\sigma = \tfrac{1}{2};$$

in the centre of the line it is given by

$$\sigma = \tfrac{1}{2}\frac{\alpha}{\alpha_\nu}.$$

Again, $F_\nu \to G_\nu$ as ν departs from ν_0. But the intensity-ratio is now given by

$$\frac{F_\nu}{G_\nu} = \frac{1+\tfrac{1}{2}b_\nu(\alpha/\alpha_\nu)}{1+\tfrac{1}{2}b_\nu},$$

instead of by a Schuster formula.

The important thing to notice is that the "number of molecules in the line of sight" is now irrelevant. The intensity-ratio depends only on b_ν, *i.e.* on the temperature gradient, and on the ratio of the selective coefficient to the general. The emergent intensity F_ν varies with ν as a linear function of $1/\alpha_\nu$, and thus the observable width would be practically the intrinsic half-width. We have of course assumed α/α_ν to be a constant; if it is not so we must take an average value.

The fundamental objection to this theory of Fraunhofer lines in its crude form—an objection brought forward years ago by Schwarzschild † in a famous paper—is that it implies that all lines should disappear as we approach the limb of the sun. For at the limb the continuous

* It should be remembered that b_ν is not arbitrary—it depends simply on the temperature distribution. It is of the order of magnitude of 2. In radiative equilibrium, for instance, the integrated intensity $B(=acT^4/4\pi)$ is given on the present approximation by $B=B_0(1+2\sigma)$, and consequently $2B_0=\int(B_\nu)_0 b_\nu d\nu$. Thus the mean value of b_ν, taken with $(B_\nu)_0$ as a weighting factor, is 2.

† *Berlin Sitz.*, 1914, p. 1198.

background must also be that corresponding to the boundary temperature. (If we suppose there are different "boundaries" for the atoms producing the general absorption and those producing the selective, with different temperatures, then we get back to the other theory.) For this reason Schwarzschild concluded that the lines were caused by "selective scattering," or, as I prefer to say, local radiative equilibria.* Considerations of temperature are then irrelevant. In my chromosphere paper I showed in detail that this gave a consistent picture of the phenomena concerned.

The purely thermodynamic theory labours under the disadvantage that it assumes that at each level there is a definite temperature. But "temperature" may be used in many senses, and I showed that the "temperature" judged from the speeds of the atoms at high levels was not necessarily that judged from their allocation amongst different stationary states. Moreover, the allocations of the atoms of different elements at the same level will not necessarily correspond to the same temperature. I have shown † that when a line is produced by transitions from a state 1 to a state 2, if n_1 and n_2 are the numbers of atoms per cm.^{3} in the two states, then the intensity of radiation characteristic of this allocation is

$$\frac{2h\nu^3}{c^2} \cdot \frac{n_2}{n_1 - n_2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

When the atoms are in thermodynamic equilibrium at temperature T, we have

$$n_2/n_1 = e^{-h\nu/kT},$$

and (13) reduces to

$$\frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1},$$

which is simply Planck's formula for $B_\nu(T)$, the intensity of black-body radiation at temperature T. But when no unique T exists we are thrown back on the more fundamental formula (13). The value of the emergent intensity must be deduced from (13) by integration inwards through the layers in the usual way, allowing for absorption.

It is not that the thermodynamic treatment is wrong—it is simply that it is inapplicable. It is probable (or so it appears to me) that as we go inwards from the high-level chromosphere to the photosphere, the separate radiative equilibria at high levels gradually shade off into practically thermodynamic equilibrium in the lower reversing layer. The two theories join hands. For each line the residual central intensity depends on the limiting value of n_2/n_1 at the highest levels concerned. In all cases where there is a definite temperature at these levels, the residual intensity calculated from the values of n_2/n_1 will agree with that calculated from the simple thermodynamic theory. This will nearly always be the case for subordinate lines, which necessarily originate at considerable depths as compared with principal lines.‡ But the dependence on n_2/n_1 instead of on T provides just that flexibility which

* *M.N.*, **85**, 136, 1925. "Local," in the sense of "each confined to a separate frequency ν."

† *M.N.*, *ibid.*

‡ "Ultimate lines," to adopt Russell's term, *Astrophys. Journ.*, **61**, 225, 1925.

will permit different residual intensities in different lines, according to the atomic circumstances. We must calculate the emergent intensity, in fact, directly from the atoms concerned in producing the line in question, and not from a derived statistical variable, the temperature, which indeed may not happen to exist. The flexibility just mentioned permits in effect our being able to take different values of T at one and the same level. The possibility of independent behaviours of the different atomic species at the same level is closely bound up with the independent effects of selective radiation-pressure in sustaining them against gravity.

5. As Professor Stewart refers to my using "a particular theory of scattering," * it may be of interest to add a few words on the relation of the theory I use † to the classical electromagnetic theory. The main formulæ of the theory are

$$a\Delta\nu=\frac{B_{1\to 2}h\nu_0}{4\pi}=\frac{c^2}{8\pi\tau\nu_0^2}, \quad \mathbf{F}_\nu=\frac{8\nu_0^2\tau mg}{c}.$$

These rest on considerations of the utmost generality. The former is an immediate consequence of Einstein's analysis of the thermodynamic equilibrium existing between atoms and radiation in an enclosure. The analysis assumes nothing whatever about the nature of radiation (not even that it exists in quanta) or the details of the process of radiating, save only that each process (absorption and emission) is associated with a transition between stationary states. The second formula introduces in addition the momentum of radiation. Any theory of radiation whatever must conform to these relations, in view of the evidence for the actual existence of stationary states.

Professor Stewart has pointed out that the second formula can be deduced from the classical theory, as applied to a linear oscillator, when a suitable interpretation is given to τ. But an atom is not a linear oscillator, and it would seem preferable to base the formula on such general considerations as those of thermodynamics, rather than on one particular detailed theory of radiation—the classical theory—which in so many cases gives results in contradiction with observation. In the new development of radiation theory due, in the first instance, to Bohr, Kramers, and Slater, *Phil. Mag.*, **47**, 785, 1924, and continued

* Professor Stewart uses the term "scattering" where I should use "absorption." It is true that in chromospheric equilibrium the quanta of radiation are re-emitted soon after being absorbed, and that in consequence the equations are formally the same as for scattering. But there is an essential difference between the two processes. When an atom absorbs a quantum, the energy is potentially capable of being converted into heat (by a collision of the second kind), and, moreover, there is no phase-relation between the incident and emitted light. The energy is imprisoned for a time τ. When an atom scatters a quantum, the energy is simply deflected, and there is a definite phase-relation between the incident and the scattered light. The one involves a temporary change in the structure of the atom, the other not. The usage I am advocating is that of the school of Bohr.

† I borrowed from Tolman, *loc. cit.*, merely the refined treatment which leads to $\int a d\nu$ instead of $a\Delta\nu$. I had used substantially the same theory earlier in *Phil. Mag.*, **47**, 209, 1924; *Proc. Phys. Soc. London*, **36**, 111, 1924; and *M.N.*, **84**, 354, 1924 The principle is due exclusively to Einstein, *Phys. Zeit.*, **18**, 121, 1917.

by Kramers and Heissenberg, *Zeit. für Phys.*, **31**, 681, 1925, a profound reinterpretation of classical formulæ is necessary before they give results agreeing quantitatively with experiment.

6. *Summary.*—(1) The theory of the high-level chromosphere, given in a previous paper, assumes only that the high-level atoms are emitting and absorbing within a narrow region embedded in the broader Fraunhofer line. The main features of the theory (residual intensity, density-law) are independent of the intrinsic width of the line produced by the high-level atoms alone, though the actual values of the density and pressure are proportional to it.

(2) The intrinsic width of a line as given by the classical electromagnetic theory is inadequate to account for the observed widths of Fraunhofer lines. Some other cause of broadening must be present in the reversing layer, producing the equivalent of an intrinsic half-width of the order of 0·1 A.U.

(3) Whatever may be the physical origin of broadening, the number of atoms required to produce an observed solar line with a half-width of the order of 0·3 A.U. and an observed central intensity-ratio of $\frac{1}{10}$ is of the order of magnitude that can be supported by solar radiation-pressure against solar gravity. This extends to the layer (reversing-layer+chromosphere), the conclusion previously arrived at concerning the high-level chromosphere only.

(4) The contrary conclusions of Professor Stewart appear to be due to his not having taken account of the observed order of magnitude of the intensity-ratio at the centre of a solar line.

(5) The central intensity-ratio, and the complete contour, of a Fraunhofer line play a fundamental part in theoretical investigations of the chromosphere and reversing layer. Opportunity may be taken to urge the pressing need of accurate measures of the contours of absorption lines on an absolute scale, and of their change from the centre to the limb.

The Stellar Absorption Coefficient. By Professor E. A. Milne.

1. *The Cause of the General Opacity of the Outer Regions of a Star.*—The general opacity of the outer regions of the sun has been attributed by Russell and Stewart * to the scattering and absorption of radiation by free electrons and to the " Rayleigh " scattering by bound electrons. On the other hand, the general opacity of the inner regions of a star has been attributed by Eddington † to the emission of photo-electrons by atoms, which thus become ionised in virtue of the absorption of radiation.

The calculations of Russell and Stewart were based on the classical electromagnetic theory.‡ It is probable, however, that the scattering

* *Astrophys. Journ.*, **59**, 199, 1924.
† *M.N., R.A.S.*, **83**, 32, 1922; **84**, 104, 1924.
‡ Stewart, *Nature*, **111**, 186, 1923; *Phys. Rev.*, **22**, 325, 1923.

of radiation by electrons is of the kind considered by Compton, whereby a single quantum is scattered at each "collision" of a quantum and a free electron; or at least that the laws of scattering by electrons are formally the same as if this occurred. The statistical mechanics of this mode of scattering has been worked out by Pauli *; it is shown to lead to the distribution of radiant energy given by Planck's law, whilst interaction of free electrons and radiation on the classical theory leads to a distribution in which all the energy resides in the very short wave-lengths.† For these reasons the calculations of Russell and Stewart would appear to require revision. I have suggested elsewhere ‡ the possibility that the opacity of the outer regions of a star may be accounted for by photo-electric emissions, so that the opacity of the outer layers would originate in the same way as that of the inner regions.

To test this it is necessary to deduce an expression for the absorption coefficient of an ionised gas at a high temperature from the work of Kramers § on the X-ray absorption coefficients. This has already been done by Professor Eddington, but as the circumstances in the outer layers of a star (where the number of free electrons is comparable with the number of atoms, and the ionisation potentials are small) are somewhat different from those in the interior (where the number of free electrons is large compared with the number of atoms, and the ionisation potentials are large), it seemed desirable to give an independent deduction of the formula.

The method I have used for summing over the atoms in different states of excitation and different stages of ionisation differs from Eddington's. The formula I arrived at differed from Eddington's only in a certain factor, but the interesting fact appeared that it gave definitely larger values than those given by Eddington's formula. The additional factor could be as much as 3 in favourable cases. It arose from the embodiment in the analysis of Kramers' rule for partitioning the classical spectrum amongst the quantum emissions. As Eddington has laid stress on the fact that on his calculation Kramers' theory gives a value of the stellar absorption coefficient only $\frac{1}{8}$ of that deduced from the astronomical data provided by Capella, it seemed of importance to investigate the matter in detail. Accordingly the present paper is concerned exclusively with the calculation of the absorption coefficient. The succeeding paper contains the application to the photospheric layers.

2. *The Theoretical Value of the Atomic Absorption Coefficient of an Ionised Gas.*—(a) *The Part Due to "Free-free" Electronic Transitions.*—The principle underlying both Eddington's and Kramers' calculation of an absorption coefficient is that in thermodynamic equilibrium the absorptions are balanced by the emissions. The total emission is determined by Kramers by an application of Bohr's correspondence principle. In an assembly consisting of free electrons and ionised atoms, emissions occur (a) when a free electron encountering an atom (neutral or ionised) jumps from one hyperbolic orbit to another of

* *Zeit. für Phys.*, **18**, 272, 1923.
† Jeans, *Phil. Mag.*, 1914 Jan., p. 14.
‡ *Proc. Phys. Soc. London*, **36**, 101, 1924.
§ *Phil. Mag.*, **46**, 836, 1923.

smaller energy,* (*b*) when a free electron encountering an ionised atom is captured in one of the energy-levels of the atom. Eddington has enumerated the other possibilities of emission, but the present discussion is confined to these two. The calculation for (*a*) is much simpler than for (*b*).

To obtain (*a*), let us suppose there are N_e free electrons and N atoms of a given atomic number Z per unit volume, at temperature T. Let $a_1(\nu)$ be the part of the atomic absorption coefficient for radiation of frequency ν, originating from "free-free" transitions. Then the total emission of $(\nu, \nu+d\nu)$ radiation per unit volume per second is

$$4\pi N a_1(\nu) I_\nu d\nu,$$

where

$$I_\nu = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT}-1},$$

c being the velocity of light, h Planck's constant, and k Boltzmann's gas-constant.

The number of electrons with velocities lying within the range $(v, v+dv)$ encountering a given atom so as to pass within a distance lying in the range $(p, p+dp)$, per second, is

$$8\pi^2 N_e \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{1}{2}mv^2/kT} v^3 dv\, p dp,$$

m being the mass of the electron. The energy emitted classically in the range $(\nu, \nu+d\nu)$ in such an encounter, according to Kramers, is

$$\frac{4\pi^2 Z^2 e^6}{c^3 m^2 p^2 v^2} P(\gamma) d\nu \qquad . \quad . \quad . \quad . \quad (1)$$

where

$$\gamma = 2\pi\nu m^2 p^3 v^3 / Z^2 e^4,$$

and P is a function such that

$$\int_0^\infty \frac{P(\gamma) d\gamma}{\gamma} = \frac{4}{\pi\sqrt{3}}.$$

(In this e denotes the electronic charge.) By the correspondence principle, the same amount of energy will be emitted on the average over a large number of encounters, in quanta of amount $h\nu$. Equating the total absorption to the total emission we have †

$$\frac{4\pi N a_1(\nu) d\nu}{e^{h\nu/kT}-1} \cdot \frac{2h\nu^3}{c^2} = 8\pi^2 N N_e \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \frac{4\pi^2 Z^2 e^6}{c^3 m^2} d\nu \int_{v_0}^\infty e^{-\frac{1}{2}mv^2/kT} v dv \int_0^\infty P(\gamma) \frac{dp}{p},$$

* Sometimes called "partial captures."

† A more detailed investigation, in which we determine the *probability* of an emission $h\nu$ during an encounter and insist that this probability must be independent of the statistical variable T, shows that there is an additional term in the emission proportional to I_ν, the "stimulated emission." Eddington has pointed out that as it stands this appears to make the integrated emission infinite (*M.N.*, **84**, 116, 1924). This absurdity appears to me to be a consequence of the approximations made in Kramers' deduction of (1), approximations which lead to large errors at low frequencies.

where

$$\tfrac{1}{2}mv_0^2=h\nu.$$

This leads to

$$\frac{2h\nu^3}{c^2}\cdot\frac{a_1(\nu)}{e^{h\nu/k\mathrm{T}}-1}=\frac{32\pi^2}{3\sqrt{3}}\cdot\frac{\mathrm{Z}^2e^6}{c^3(2\pi m)^{\frac{3}{2}}}\cdot\frac{e^{-h\nu/k\mathrm{T}}}{(k\mathrm{T})^{\frac{1}{2}}}\mathrm{N}_e \quad . \quad . \quad (2)$$

Equation (2) determines the absorption coefficient $a_1(\nu)$. Integrating over all frequencies * we have for the mean absorption coefficient a_1

$$a_1=\int_0^\infty a_1(\nu)\mathrm{I}_\nu d\nu\Big/\int_0^\infty \mathrm{I}_\nu d\nu=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{\mathrm{Z}^2e^6h^2}{c(2\pi m)^{\frac{3}{2}}}\cdot\frac{\mathrm{P}_e}{(k\mathrm{T})^{\frac{3}{2}}} \quad . \quad (3)$$

on inserting the partial pressure P_e of free electrons,

$$\mathrm{P}_e=\mathrm{N}_e k\mathrm{T}.$$

Formula (3) is identical with that obtained by Eddington †

2. (b) *The Part Due to Photo-electric Emissions.*—Kramers has calculated the probability of capture of an electron by a bare nucleus by an application of the correspondence principle, and the resulting absorption coefficients are in fair agreement with the resulting X-ray absorption coefficients. It is necessary to adapt Kramers' result so as to give the probability of capture of an electron in an optical level of an atom.

By an "optical level" of an atom is meant a level occupied by the "outermost" (most lightly bound) electron ‡ of a neutral, singly or multiply-ionised atom, in contradistinction to an X-ray level (which is a level occupied by one of the inner electrons), although the frequencies concerned will not necessarily lie in the optical region. An X-ray level becomes an optical level when the atom is ionised down to this level. In the interior of a star the frequencies associated with the optical levels of the atoms in their highly-ionised states are in fact of the order of X-ray frequencies. In the absorption and emission of radiation by atoms which are members of a gaseous assembly in thermodynamic equilibrium it is almost exclusively the optical levels which are concerned.

Let χ be an optical energy-level of a given atom in a given state of ionisation, q the corresponding weight, q' the weight of the lowest state of the same atom one stage more ionised. Beyond the series limit χ/h belonging to this level there will be a region of continuous absorption, arising from the ejection of the electron from the χ-level. Let ν be a frequency in this region ($\nu>\chi/h$). I have shown § that the

* Rosseland has pointed out that a different method of averaging the absorption coefficient through the spectrum ought properly to be employed when the resulting average is to be used in the theory of the radiative equilibrium of a star (*M.N.*, **84**, 524, 1924). But the theoretically correct method does not lend itself easily to the final evaluation of the mean value of the coefficient. All the results of the present paper are liable to a correction in Rosseland's sense. This tends to increase again the discrepancy between theory and observation.

† *M.N.*, **84**, 106, 1924, equation (7).

‡ Known as the "series-electron."

§ *Phil. Mag.*, **47**, 209, 1924. The presence of the additional factor q' in the numerator has been shown by the author to be necessary, in a note recently communicated to the *Philosophical Magazine.*

atomic absorption coefficient $\alpha_0(\nu)$ arising in this manner is given by the relations

$$\alpha_0(\nu)=\frac{\psi(\nu)h\nu}{4\pi}=\frac{\pi m^2\sigma q' v^2 \mathrm{F}(v)}{q}\cdot\left(\frac{c}{h\nu}\right)^2. \quad . \quad . \quad . \quad (4)$$

Here σ is the symmetry number of the atom in the given state of ionisation; v is the velocity of ejection of the photo-electron, given by

$$\tfrac{1}{2}mv^2=h\nu-\chi \quad . \quad . \quad . \quad . \quad (5)$$

$\psi(\nu)$ is a probability coefficient analogous to Einstein's probability coefficients; and $\mathrm{F}(v)$ is a function specifying the probability of electron-capture.

The function $\mathrm{F}(v)$ is equivalent to Kramers' "β," the effective cross-section for electronic encounters resulting in capture in the χ-level. If s is the target radius, in Eddington's terminology, then the exact relations are

$$\beta=\pi s^2=2\pi \mathrm{F}(v) \quad . \quad . \quad . \quad . \quad (6)$$

From an application of the correspondence principle, under certain restrictions into which we need not enter, Kramers has found * that

$$\beta=g'\frac{32\pi^2\mathrm{Z}^2e^6}{3\sqrt{3}c^3m^2h\nu v^2}\Delta\nu \quad . \quad . \quad . \quad (7)$$

In this, g' is a numerical coefficient not very different from unity. The coefficient $\Delta\nu$ is a frequency interval surrounding the frequency ν with the following meaning: on the classical theory an electron encountering the atom would emit a continuous spectrum, and it is supposed that on the average the portion of this spectrum lying in the interval $\Delta\nu$ is the energy actually emitted, as monochromatic radiation, when the electron is captured in the χ-level.

In the capture of an electron by a bare nucleus, χ is equal to $2\pi^2\mathrm{Z}^2e^4m/n^2h^2$, where n is the total quantum number of the χ-level. Kramers required a partition of the spectrum into intervals surrounding the frequencies capable of being emitted by the electron, namely frequencies corresponding to capture in the levels of total quantum number 1, 2, . . . n . . ., and he chose for $\Delta\nu$ the expression

$$\Delta\nu=\frac{1}{h}\cdot\frac{2\pi^2\mathrm{Z}^2e^4m}{h^2}\left[\frac{1}{(n-\frac{1}{2})^2}-\frac{1}{(n+\frac{1}{2})^2}\right] \quad . \quad . \quad (8)$$

Formulæ (4), (6), and (7) combined with this value of $\Delta\nu$ then gave an expression † for the X-ray absorption coefficient which agrees with the observed values if g' is taken as 2·2 or 2·4.

We have to extend this definition of $\Delta\nu$ to optical levels. The in-

* *Phil. Mag.*, **46**, 857, 1923. I quote his formula (56) after replacing the factor $4\pi^2\mathrm{N}^2e^4m/h^3n^3$ by the factor $\Delta\nu$, from which it originated according to his formula (53).

† Actually, Kramers took the [. . .] on the right-hand of (8) as being approximately $2/n^3$. If, for $n=1$, he had taken the actual value $\frac{32}{9}$, instead of his approximation 2, his g' would have been 1·3. This is made use of in § 6 below.

terval $\Delta\nu$ given by (8) is the range of frequency corresponding to a range of $\pm\frac{1}{2}$ in total quantum number on either side of the level concerned. Now in general a set of optical levels $\chi^{(s)}$, $s=0, 1, 2 \ldots$ can be represented by a Rydberg formula,*

$$\chi^{(s)}=\frac{\text{const.}}{(n+s-Q)^2},$$

where n is the total quantum number for the lowest state $\chi^{(0)}$ and Q is the "quantum defect." Writing this in the form

$$\chi^{(s)}=\frac{\chi^{(0)}(n-Q)^2}{(n+s-Q)^2},$$

we can follow Kramers by adopting for $\Delta\nu_s$, the interval surrounding the frequency emitted by an electron falling into the level $\chi^{(s)}$, a formula

$$\begin{aligned}\Delta\nu_s&=\frac{\chi^{(0)}(n-Q)^2}{h}\left[\frac{1}{(n+s-\frac{1}{2}-Q)^2}-\frac{1}{(n+s+\frac{1}{2}-Q)^2}\right]\\&=\frac{\chi^{(0)}(n-Q)^2}{h}\Delta(s), \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)\end{aligned}$$

say. Combining the various formulæ we have now (omitting g')

$$a_0(\nu)=\frac{16\pi^2}{3\sqrt{3}}\cdot\frac{Z^2e^6}{ch}\cdot\frac{q'\sigma}{q}\cdot\frac{\chi^{(0)}}{(h\nu)^3}\cdot(n-Q)^2\Delta(s) \quad . \quad . \quad (10)$$

(On the hypothesis of capture by nuclear collisions put forward by Eddington in his first paper on the stellar absorption coefficient, the corresponding probability-capture function $F_c(v)$ is given by

$$2\pi F_c(v)=\pi s^2=\frac{\pi e^4}{m^2c^2v^2}Z\left(Z+\frac{2d}{b}\right) \quad . \quad . \quad (11)$$

where d is the radius of the nucleus and b is e^2/mc^2. For a nucleus of the size of an electron, $d=\frac{2}{3}e^2/mc^2$, and $2d/b$ is $4/3$; save for hydrogen, $2d/b$ may be neglected in comparison with Z. The corresponding value $a_c(\nu)$ for the atomic absorption coefficient is then found to be

$$a_c(\nu)=\tfrac{1}{2}\pi e^4Z\left(Z+\frac{2d}{b}\right)\frac{q'\sigma}{q}\cdot\frac{1}{(h\nu)^2} \quad . \quad . \quad . \quad (12)$$

The ratio of expressions (10) and (12) is, approximately,

$$\frac{a_0(\nu)}{a_c(\nu)}=\frac{32\pi}{3\sqrt{3}}\cdot\frac{e^2}{ch}\cdot\frac{\chi^{(0)}}{h\nu}\cdot(n-Q)^2\Delta(s) \quad . \quad . \quad (13)$$

Now with $e=4{\cdot}774\times10^{-10}$, $c=3\times10^{10}$, $h=6{\cdot}55\times10^{-27}$ we have $e^2/ch=1{\cdot}16\times10^{-3}$ and

$$\frac{a_0(\nu)}{a_c(\nu)}=2{\cdot}24\times10^{-2}\frac{\chi^{(0)}}{h\nu}(n-Q)^2\Delta(s) \quad . \quad . \quad . \quad (14)$$

* D. R. Hartree, *Proc. Camb. Phil. Soc.*, **22**, 467, 1924. We employ Q for the quantum defect instead of his q, to avoid confusion with our use of q for a weight.

For capture in the lowest level of a bare nucleus, for example, we have $n=1$, $Q=0$, $s=0$, and thus for the frequency corresponding to the series limit, $h\nu=\chi^{(0)}$, we have

$$\frac{a_0(\nu)}{a_e(\nu)}=0{\cdot}08.$$

Thus for the monochromatic absorption coefficient at the series limit, Kramers' formula gives a value about $\frac{1}{12}$ of that given by Eddington's capture formula.)

Formula (10) may be used to estimate monochromatic absorption coefficients in the region beyond a series limit.* Our immediate object, however, is to deduce the value of the mean absorption coefficient for integrated radiation for an assembly of atoms in thermodynamic equilibrium at temperature T. We follow the procedure already adopted for the same purpose in a former paper.†

We begin by fixing attention on atoms in a given state of ionisation, say atoms which have lost $r-1$ electrons, where r takes the values 1, 2 . . . for the successive states of ionisation. The weights of the successive levels $\chi_r^{(0)}$, $\chi_r^{(1)}$ we denote by $q_r^{(0)}$, $q_r^{(1)}$ . . . . Of the whole number of atoms, let there be a fraction $x_r^{(s)}$ which are $(r-1)$-fold ionised, and in the state $\chi_r^{(s)}$. Temporarily we shall now omit the suffix r for convenience, so long as we are dealing with atoms in the same stage of ionisation. Then the contribution of the $(r-1)$-fold ionised atoms to the atomic absorption coefficient is $a_0(r, \nu)$, say, given by

$$a_0(r,\nu)=\sum_{s=0} x^{(s)}a_0^{(s)}(\nu)=\frac{16\pi^2}{3\sqrt{3}}\cdot\frac{Z^2e^6\sigma q'\chi^{(0)}(n-Q)^2}{ch(h\nu)^3}\sum_{s=0}\frac{x^{(s)}}{q^{(s)}}\Delta(s) \qquad (15)$$

Their contribution to the mean absorption coefficient for integrated radiation is accordingly

$$a_0(r)=\sum_{s=0} x^{(s)}\int_{\nu(s)}^{\infty} a_0^{(s)}(\nu)I_\nu d\nu \Big/ \int_0^\infty I_\nu d\nu \qquad . \qquad . \qquad (16)$$

where

$$h\nu^{(s)}=\chi^{(s)}.$$

On making the substitutions

$$y=\frac{h\nu}{kT},\quad y^{(s)}=\frac{h\nu^{(s)}}{kT}=\frac{\chi^{(s)}}{kT},$$

(16) becomes

$$a_0(r)=\frac{16\pi^2Z^2e^6\sigma q'\chi^{(0)}(n-Q)^2}{3\sqrt{3}ch(kT)^3}\sum_{s=0}\frac{x^{(s)}}{q^{(s)}}\Delta(s)\int_{y(s)}^{\infty}\frac{dy}{e^y-1}\Big/\int_0^\infty\frac{y^3dy}{e^y-1} \qquad (17)$$

Now

$$\int_0^\infty\frac{y^3dy}{e^y-1}=\frac{\pi^4}{15},$$

and

$$\int_{y(s)}^{\infty}\frac{dy}{e^y-1}=-\log\,(1-e^{-y^{(s)}}).$$

* See § 2 of the succeeding paper.

† *Phil. Mag.*, **47**, 231, 1924.

Further, in thermodynamic equilibrium we have by Boltzmann's principle

$$\frac{x^{(s)}}{x^{(0)}}=\frac{q^{(s)}}{q^{(0)}}e^{-(y^{(0)}-y^{(s)})}.$$

Hence

$$a_0(r)=-\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6q'\sigma e^{-y^{(0)}}x^{(0)}}{ch(kT)^3q^{(0)}}\sum_{s=0}\chi^{(0)}(n-Q)^2\Delta(s)e^{y^{(s)}}\log(1-e^{-y^{(s)}}) \quad (18)$$

To effect the summation, we observe that from (9)

$$\chi^{(0)}(n-Q)^2\Delta(s)=\chi^{(s-\frac{1}{2})}-\chi^{(s+\frac{1}{2})}=kT(y^{(s-\frac{1}{2})}-y^{(s+\frac{1}{2})})=kT\Delta y^{(s)},$$

say, where $\chi^{(s-\frac{1}{2})}$ . . . are the values which would be given by the Rydberg formula for $\chi^{(s)}$ on replacing s by $(s-\frac{1}{2})$. . . . Thus the summation in (18) can be written

$$kT\sum_{s=0}e^{y^{(s)}}\log\left(\frac{1}{1-e^{-y^{(s)}}}\right)\Delta y^{(s)} \quad . \quad . \quad . \quad (19)$$

Now the intervals Δy completely fill up the interval from $s=-\frac{1}{2}$ to $s=\infty$. Hence an approximation to (19) at least as valid as the approximations made by Kramers is obtained by replacing the summation by the integral to which it is an approximative sum, in the form

$$kT\int_0^Y e^y\log\frac{1}{1-e^{-y}}dy \quad . \quad . \quad . \quad . \quad (20)$$

where

$$Y=y^{(-\frac{1}{2})}=\frac{\chi^{(0)}}{kT}\cdot\frac{(n-Q)^2}{(n-\frac{1}{2}-Q)^2}.$$

The great advantage of (20) is that it is independent of the precise mode of partition of the classical spectrum amongst the different quantum frequencies, provided only that the classical spectrum is cut off at a level half a unit quantum number lower than $\chi^{(0)}$.

The integral in (20) is easily evaluated, and comes to

$$Y-(e^Y-1)\log(1-e^{-Y}).$$

Expanding in powers of e^{-Y}, this is equal to

$$1+Y-e^{-Y}\left(\frac{1}{1\cdot2}+\frac{e^{-Y}}{2\cdot3}+\frac{e^{-2Y}}{3\cdot4}+\ldots\right).$$

In practice, e^{-Y} is negligible compared with unity.

Formula (18) thus simplifies to

$$a_0(r)=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6}{ch}\cdot\frac{1}{(kT)^2}\cdot\frac{\sigma q'x^{(0)}e^{-y^{(0)}}}{q^{(0)}}\cdot\left(1+\frac{\chi^{(0)}}{kT}\cdot\frac{(n-Q)^2}{(n-\frac{1}{2}-Q)^2}\right) \quad (21)$$

It may be observed that we have calculated as though the atom possessed only one series of levels. But we have paid strict attention to the weights of the levels, *i.e.* we have multiplied in each case by a factor corresponding to the different number of levels into which a simple series of levels of the hydrogen type is split up for more complicated atoms. A little consideration shows that, apart from a certain

amount of tacit averaging, our procedure has taken strict account of *all* the optical levels of the atom in the given state of ionisation.

We now restore the suffixes denoting the degree of ionisation. The quantity $\chi^{(0)}$, since it is the lowest level of the $(r-1)$-fold ionised atoms, is simply χ_r, the rth ionisation potential; $x^{(0)}$ is now written x_r, the fraction of the atoms which are $(r-1)$-fold ionised and in their lowest state; $q^{(0)}$, their weight, is written q_r, and q' is simply q_{r+1}. The mean absorption coefficient a_0, account being taken of all the atoms in their proper stages of ionisation, is now seen to be given by

$$a_0=\sum_{r=1} a_0(r)=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6}{ch}\cdot\frac{1}{(kT)^2}\cdot\sum_{r=1}\frac{\sigma_r q_{r+1}x_r e^{-y_r}}{q_r}\left(1+y_r\frac{(n_r-Q_r)^2}{(h_r-\frac{1}{2}-Q_r)^2}\right) \quad (22)$$

where

$$y_r=\chi_r/kT.$$

Consecutive fractions x_r, x_{r+1} are connected by the ionisation formula *

$$\frac{x_{r+1}}{x_r}P_e=\frac{q_{r+1}}{q_r}e^{-y_r}(kT)^{\frac{3}{2}}\frac{(2\pi m)^{\frac{3}{2}}\sigma_r}{h^3} \quad . \quad . \quad . \quad (23)$$

By means of this we can effect a considerable simplification by substituting for

$$\frac{x_r\sigma_r q_{r+1}}{q_r}e^{-y_r} \quad . \quad . \quad . \quad . \quad (23a)$$

in (22). We find then as the formula for the part of the stellar atomic absorption coefficient depending on photo-electric absorptions with ejections of electrons

$$a_0=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6h^2}{c(2\pi m)^{\frac{3}{2}}}\cdot\frac{P_e}{(kT)^{\frac{5}{2}}}\cdot\sum_{r=1}x_{r+1}\left(1+\frac{\chi_r}{kT}\cdot\frac{(n_r-Q_r)^2}{(n_r-\frac{1}{2}-Q_r)^2}\right) \quad (24)$$

3. *The Final Formula for the Stellar Absorption Coefficient.*—Since all but a very small fraction of the atoms will be in their lowest states † and therefore $\sum_{r=1}x_r=1$, we may replace $\sum_{r=1}x_{r+1}=\left(\sum_{r=1}x_r\right)-x_1$ by unity, x_1 being negligible in the interior of a star.

* This is the correct formula, when x_r and x_{r+1} have strictly the meanings stated, *i.e.* when they represent the fractions of atoms which are $(r-1)$-fold and r-fold ionised respectively, *and are in their lowest states.* No factors $b_r(T)$, $b_{r+1}(T)$ occur, since when these are present the fractions x_r, x_{r+1} represent the *total* ionised fractions, counting all the stationary states. This form of the formula is the subject of a note by the writer, which has been communicated to the *Philosophical Magazine.* In this note it is pointed out that the formula used by R. H. Fowler and E. A. Milne in *M.N.*, **83**, 407, 1923 (and thence unfortunately frequently quoted by others) contains an error. A term $+\log b'(T)$ should be added to the right-hand side of equation (1) in that paper, where $b'(T)$ is the partition function for the ionised atoms.

† If an ion has a number of levels close together and approximately the same as the lowest, this might appear to involve an error. But the deduction of (24), resting as it does fundamentally on (4), assumes that when an electron is ejected the extra-ionised remainder is always in its lowest state. Actually, a correction for the possibility of its being in higher states is required in all the formulæ. But it would make the notation very troublesome. The right result is obtained if we regard any group of levels close together as forming a single level.

We now add in the part a_1 corresponding to "free-free" transitions as given by (3), and obtain for the total coefficient

$$a=a_0+a_1=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6h^2}{c(2\pi m)^{\frac{3}{2}}}\cdot\frac{P_e}{(kT)^{\frac{5}{2}}}\cdot\left(2+\sum_{r=1}x_{r+1}\frac{\chi_r}{kT}\cdot\frac{(n_r-Q_r)^2}{(n_r-\frac{1}{2}-Q_r)^2}\right) \quad (25)$$

This is the final expression for the atomic absorption coefficient for a highly-ionised gas. We have taken account of the presence, in its due proportion, of every atomic level in every stage of ionisation, the contribution of each level being reckoned for the spectral range in which it is actually absorbent. The analysis may have appeared complicated, but it can hardly be said to be artificial; there has been a natural momentum in the trend of the analysis, whereby at each stage the new factors introduced have merged with the existing ones so as to produce simplifications. This is especially so in the appearance of a factor (23*a*) which is wiped out by the ionisation formula. It may be observed, with respect to the summation in (25), that χ_r, the ionisation potential of atoms present to the amount x_r, is to be multiplied by the *next succeeding* fraction x_{r+1}.

4. *Comparison with Eddington's Formula.*—Eddington's value of the atomic absorption coefficient as deduced by him from Kramers' theory by a somewhat different train of argument, is given by formula (7) in his second paper on the stellar absorption coefficient. Actually his (7) as it stands gives the part corresponding to free-free transitions; it is identical with our (3). For the part arising from photo-electric emissions he multiplies this by a certain factor $h\nu_1/kT$. In effect he chooses for ν_1 a value $\nu_1=\chi/h$, where χ may be regarded as the dominant ionisation potential under the conditions P_e, T. When the successive ionisation potentials of an atom are well-separated, only two states of ionisation are present in appreciable proportions at the same time. Eddington therefore chooses for χ the value which would make x_{r+1}/x_r, the ratio of the numbers of atoms in the two dominant stages of ionisation, equal to unity, as given by our equation (23). Thus he takes $x_r=x_{r+1}$, ($=\frac{1}{2}$ approximately), as defining χ. Translated into our notation, his formula for the total atomic absorption coefficient, a', is found to reduce to

$$a'=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2h^2e^6}{c(2\pi m)^{\frac{3}{2}}}\cdot\frac{P_e}{(kT)^{\frac{5}{2}}}\left(1+\frac{\chi}{kT}\right) \quad . \quad . \quad (26)$$

The ratio of ours to Eddington's is

$$\frac{a}{a'}=\frac{2+\sum_{r=1}x_{r+1}\frac{\chi_r}{kT}\frac{(n_r-Q_r)^2}{(n_r-\frac{1}{2}-Q_r)^2}}{1+\chi/kT} \quad . \quad . \quad . \quad (27)$$

Though the steps in our process of averaging are very different from Eddington's, the formulæ only differ in their final factor, of zero dimensions. This final factor differs in the two formulæ in two respects. In the first place our summation replaces Eddington's mean, and as a consequence of the more detailed integration there is an extra term equal to unity; but the expression $\Sigma x_{r+1}\chi_r$ will not differ much from

Eddington's χ, for $\Sigma x=1$ and so $\Sigma x_{r+1}\chi_r$ is simply a weighted mean. But in the second place, in our formula the ionisation potentials are multiplied by factors

$$\frac{(n_r-Q_r)^2}{(n_r-\frac{1}{2}-Q_r)^2} \quad . \quad . \quad . \quad . \quad (28)$$

which have no place in Eddington's formula.

The presence of these factors is an essential consequence of our train of argument. They represent simply Kramers' partition of the classical spectrum. When the series-electron occupies a K-orbit, $n_r=1$ and $Q_r=0$, and the factor is equal to 4. When the series-electron occupies an L-orbit and the K-orbits are occupied, $n_r=2$, and provided the atomic number is not very low, Q_r is practically zero again; the factor is now equal to $16/9$. The factor 4 simply corresponds to the fact that on Kramers' theory an electron of energy $h\nu_0$, which can emit various frequencies on capture up to $\nu=\nu_0+\nu_K$ (when $h\nu_K$ is the energy of the K-orbit), is supposed actually to emit in a large number of captures the energy in the classical spectrum up to $\nu=\nu_0+4\nu_K$. For capture in an L-orbit, the argument involving the partitioning of the spectrum by half-quantum numbers gives $\nu_0+\frac{16}{9}\nu_L$ as the limit. A cruder argument of the same character would give the limit as $\nu_0+\nu_L+\frac{1}{2}(\nu_K-\nu_L)$, which gives $\nu_0+\frac{5}{2}\nu_L$, since $\nu_K=4\nu_L$ approximately. As the method adopted by Kramers for partitioning the spectrum is entirely arbitrary, little stress can be laid on the exact numbers 4, $\frac{16}{9}$. In fact they are very sensitive to small changes in the formal description of the partition; *e.g.* if we used a range of quantum number $(-\frac{3}{4}, +\frac{1}{4})$, instead of $(-\frac{1}{2}, \frac{1}{2})$, the numbers would be 16 and $64/25$. We shall adopt a factor 4 for the K-orbits and a factor 2 for the L-orbits. We shall refer to these briefly as the "Kramers factors."

These same considerations are stated by Eddington, yet no corresponding factors appear in his formula. Unless I misunderstand him, he explicitly leaves them to be looked after by his method of choosing χ. Actually, as we shall see, his χ is often little bigger than our $\Sigma x_{r+1}\chi_r$, so that the Kramers factors are not properly taken account of.

Eddington gives for the ratio of the part of the absorption coefficient due to emissions of electrons to the part due to "free-free" transitions as

$$\frac{a_0'}{a_1'}=\frac{\chi}{kT} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

whilst our formulæ (3) and (24) give

$$\frac{a_0}{a_1}=1+\Sigma x_{r+1}\frac{\chi_r}{kT}\cdot\frac{(n_r-Q_r)^2}{(n_r-\frac{1}{2}-Q_r)^2} \quad . \quad . \quad . \quad (30)$$

and his a_1' is identical with our a_1. Formulæ (29) and (30) exhibit the difference between our formulæ very clearly.

The correction appears to me to go some way towards removing the discrepancy which Eddington has found between the theoretical value of the absorption coefficient and the astronomical value he has deduced from the mass, luminosity, and effective temperature of Capella. The theoretical value, according to him, was smaller than the astro-

nomical by a factor of $\frac{1}{8}$. Our version of the theoretical formula for the absorption coefficient appears capable, *prima facie*, of producing an additional factor 4 in favourable circumstances, thus reducing the discrepancy to a factor of $\frac{1}{2}$. We proceed to investigate the dependence of our final factor on the distribution of the ionisation potentials. It will be found that the factor is rather sensitive to the precise distribution, and always amounts to less than 4; it can, however, be 3.

5. *Dependence on Ionisation Potentials.*—If χ is Eddington's dominant ionisation potential, we have

$$1=\frac{q_{r+1}\sigma_r(2\ m)^{\frac{3}{2}}}{q_r h^3}\cdot\frac{(k\mathrm{T})^{\frac{5}{2}}}{\mathrm{P}_e}e^{-\chi/k\mathrm{T}} \quad . \quad . \quad . \quad (31)$$

The actual fractions x_{r+1}, x_r are, however, given by (23). Employing (31), (23) may be written

$$\frac{x_{r+1}}{x_r}=e^{-(\chi_r-\chi)/k\mathrm{T}}.$$

Formula (31) defines $(k\mathrm{T})^{\frac{5}{2}}/\mathrm{P}_e$ in terms of χ, and so apart from complications arising from differing weights and symmetry numbers we have for the adjacent stages of ionisation

$$\frac{x_{r+2}}{x_{r+1}}=e^{-(\chi_{r+1}-\chi)/k\mathrm{T}},\qquad \frac{x_r}{x_{r-1}}=e^{-(\chi_{r-1}-\chi)/k\mathrm{T}},$$

and so on. It is clear that $x_{s+1}>x_s$ so long as $\chi_s>\chi$, and that thereafter $x_{s+1}<x_s$. If χ occurs between χ_r and χ_{r-1}, the greatest x is x_r, *i.e.* the majority of the atoms have lost $r-1$ electrons.

A glance at Hartree's tables of the successive ionisation potentials of iron and silver shows that the two K-levels possess neighbouring ionisation potentials and form a group well separated from the others, and that similarly the eight L-levels possess neighbouring ionisation potentials which form a group. By "neighbouring values" in this connection we mean values which differ by an amount which is at most of the order of kT. Thus the last two ionisation potentials of iron, according to Hartree,* are 9150 volts and 8650 volts, which differ by 500 volts. But at the central temperature of Capella, $9{\cdot}56\times10^6$ degrees, kT is 822 volts, so that for rough purposes the difference between 9150 and 8600 is negligible.

Suppose then we have a group of p such ionisation potentials, and that the temperature and pressure are such that χ as given by (31) is approximately equal to them. Denote them by χ_r, χ_{r+1} . . . , χ_{r+p-1}. Then it is clear that x_r, x_{r+1} . . . , x_{r+p} are approximately equal, and the other x's negligible. Hence each is approximately $1/(p+1)$. Hence approximately

$$\sum_{s=1}x_{s+1}\chi_s=\frac{p}{p+1}\chi.$$

* Hartree points out that the energy of binding of an electron in the K, L, M . . . orbits when it is a "series-electron" is in general considerably bigger than the energies of the corresponding X-ray levels as measured in the laboratory. The absorption limit for the K-series of Fe, for example, is only 7100 volts. (*Proc. Camb. Phil. Soc.*, **22**, 474, 1924.)

Now $p=2$ for the K-levels and $p=8$ for the L-levels. We have adopted 4 and 2 as the Kramers factors corresponding to these levels. Hence the summation factors in our expression for the absorption coefficient in these two cases are

$$2+4\times\frac{2}{3}\cdot\frac{\chi}{k\mathrm{T}}, \qquad 2+2\times\frac{8}{9}\cdot\frac{\chi}{k\mathrm{T}},$$

i.e.

$$2+\frac{8}{3}\cdot\frac{\chi}{k\mathrm{T}}, \qquad 2+\frac{16}{9}\cdot\frac{\chi}{k\mathrm{T}},$$

instead of Eddington's $1+\chi/k\mathrm{T}$.

When χ is not exactly equal to the mean χ_m of the p adjacent χ's, it can be shown * that a better approximation is

$$\Sigma x_{s+1}\frac{\chi_s}{k\mathrm{T}}=\frac{p}{p+1}\left[\frac{\chi}{k\mathrm{T}}+\frac{\chi-\chi_m}{k\mathrm{T}}\left(\frac{1}{2}\frac{\chi}{k\mathrm{T}}-1\right)\right]+\frac{\chi_{r-1}}{(p+1)k\mathrm{T}},$$

which may be appreciably bigger than the previous approximation. It is clear, in fact, that if $\chi>\chi_m$, the higher values of the χ's become relatively more weighted.

If χ occurs between two well-separated χ's, say χ_r and χ_{r-1}, then approximately $x_r=1$ and the other x's are almost zero. Hence in this case the summation factor is $2+4\chi_{r-1}/k\mathrm{T}$ or $2+2\chi_{r-1}/k\mathrm{T}$, instead of $1+\chi/k\mathrm{T}$.

A final case is when χ is somewhat bigger than the last ionisation potential, which is, of course, χ_Z. In this case the atoms are almost all bare nuclei, and $x_{Z+1}=1$, the other x's being negligible. We are necessarily dealing here with the formation of a K-orbit, and so the factor is $2+4\chi_Z/k\mathrm{T}$. This formula will be valid whenever $\chi/k\mathrm{T}$ exceeds $\chi_Z/k\mathrm{T}$ by 2 or more.

We may summarise these results as follows. (In each case the summation $\Sigma x_{s+1}\chi_s/k\mathrm{T}$ must be multiplied by the appropriate Kramers factor, and 2 must be added, in order to give the summation factor comparable with Eddington's $1+\chi/k\mathrm{T}$.)

(1) $\chi_r\gg\chi\gg\chi_{r-1}$; $\quad x_r=1$; $\quad \Sigma x_{s+1}\chi_s=\chi_{r-1}$.

(2) $\chi_{r+1}\gg\chi_r\doteqdot\chi\gg\chi_{r-1}$; $\quad x_{r+1}=x_r=\frac{1}{2}$; $\quad \Sigma x_{s+1}\chi_s=\frac{1}{2}(\chi+\chi_{r-1})$.

(3) $\chi_{r+p}\gg\chi_{r+p-1}\doteqdot\chi_{r+p-2}\doteqdot\ \ldots\ \doteqdot\chi_r\doteqdot\chi\gg\chi_{r-1}$;

$x_{r+p}=x_{r+p-1}=\ \ldots\ =x_r=1/(p+1)$; $\quad \Sigma x_{s+1}\chi_s=\dfrac{p\chi}{p+1}+\dfrac{\chi_{r-1}}{p+1}$.

(4) $\chi\gg\chi_Z$; $\quad x_{Z+1}=1$; $\quad \Sigma x_{s+1}\chi_s=\chi_Z$.

It should be noticed that a mass of bare nuclei and free electrons is hardly less absorbent than a mass of atoms well furnished with bound electrons, given the same number of free electrons, though apparently at any given instant there are no electrons in position which can give rise to absorption. The explanation is, of course, that captures

* The algebra is rather tedious.

occur just as often as in any other case, and every capture must be compensated by an absorption.

The general conclusion is that for atoms whose K-ionisation potentials are a little less than the critical potential χ, our formula gives nearly three times the value given by Eddington's formula.

6. *Numerical Values.*—For the centre of Capella, assuming a mean molecular weight $\mu=2{\cdot}2$, Eddington has found

$$T=9{\cdot}56\times10^{6}, \qquad \rho=0{\cdot}1414,$$

and he showed that the astronomical data gave a mass absorption coefficient of 47·5 for this temperature and density. For this T, and the P_e corresponding to this T, ρ, and μ, Eddington found $\chi/kT=7{\cdot}505$ by applying (31). We can therefore simplify our arithmetic by employing the value

$$\frac{(kT)^{\frac{5}{2}}(2\pi m)^{\frac{3}{2}}}{P_e h^3}=\text{exp. }(7{\cdot}505)$$

in evaluating (25). We get for the atomic absorption coefficient at the centre of Capella

$$a=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6}{ch(kT)^2}\,\text{exp. }(-7{\cdot}505)\times(2+\Sigma).$$

For definiteness insert Z=26, corresponding to iron, and divide by the mass of the iron atom $56\times1{\cdot}64\times10^{-24}$ to get the mass absorption coefficient k. Inserting

$$e=4{\cdot}774\times10^{-10}, \quad e^6=1{\cdot}184\times10^{-56}, \quad h=6{\cdot}55\times10^{-27},$$
$$c=3\times10^{10}, \quad k\text{ (Boltzmann)}=1{\cdot}37\times10^{-16},$$

we have for the centre of Capella

$$k_{Fe}=0{\cdot}750\times(2+\Sigma).$$

According to Eddington's formula, we use instead of our $2+\Sigma$ the factor $1+7{\cdot}505$, and obtain

$$k'_{Fe}=6{\cdot}37.$$

(This agrees sufficiently well with the value actually computed by Eddington, namely 5·56.) We proceed to evaluate our factor $2+\Sigma_{Fe}$.

According to Hartree, the last three ionisation potentials of iron are (in volts)

$$\chi_{24}=2010, \quad \chi_{25}=8650, \quad \chi_{26}=9150.$$

At the temperature $T=9{\cdot}56\times10^{6}$, kT in volts is 822; hence since $\chi/kT=7{\cdot}505$, we have $\chi=6180$. This corresponds to case (1) above. χ lies well between χ_{24} and χ_{25}, which shows that most of the atoms retain two electrons. This position of χ is very unfavourable for a high absorption coefficient. We have

$$\chi_{24}/kT=2{\cdot}45, \quad \chi_{25}/kT=10{\cdot}53, \quad \chi_{26}/kT=11{\cdot}13,$$

and thus

$$\frac{x_{27}}{x_{26}}=e^{-3.63}; \quad \frac{x_{26}}{x_{25}}=e^{-3.03}=0{\cdot}05; \quad \frac{x_{25}}{x_{24}}=e^{5.05}.$$

This shows that $x_{25}=0{\cdot}95$, $x_{26}=0{\cdot}05$, approximately, and hence

$$2+\Sigma_{Fe}=2+4\times0{\cdot}05\times10{\cdot}53+2\times0{\cdot}95\times2{\cdot}45=8{\cdot}95.$$

Thus

$$k_{Fe}=6{\cdot}80.$$

This is practically the same as Eddington's value. We practically get the low χ_{24}/kT as the only important term in the summation factor; this is much smaller than χ/kT, but the Kramers factor brings it up again. Eddington's approximation happens to be just right here.

But suppose we go four places along the periodic table to titanium, of atomic number 22 and atomic weight 48. From Hartree's data, I calculate the last three ionisation potentials as being

$$\chi_{20}=1410,\quad \chi_{21}=6130,\quad \chi_{22}=6550.$$

Hence

$$\chi_{20}/kT=1{\cdot}72,\quad \chi_{21}/kT=7{\cdot}46,\quad \chi_{22}/kT=7{\cdot}97.$$

We find

$$\frac{x_{23}}{x_{22}}=e^{-0{\cdot}47}=0{\cdot}625,\quad \frac{x_{22}}{x_{21}}=e^{0{\cdot}04}=1{\cdot}04,\quad \frac{x_{21}}{x_{20}}=e^{5{\cdot}78},$$

whence

$$x_{23}=0{\cdot}242,\quad x_{22}=0{\cdot}388,\quad x_{21}=0{\cdot}370.$$

The summation factor is

$$2+\Sigma_{Ti}=2+4(0{\cdot}242\times7{\cdot}97+0{\cdot}388\times7{\cdot}46)+2\times0{\cdot}370\times1{\cdot}72=22{\cdot}87,$$

and the absorption coefficient, on correcting for the change of atomic number and atomic weight, is

$$k_{Ti}=0{\cdot}75\times22{\cdot}87\times\left(\frac{22}{26}\right)^2\times\frac{56}{48}=14{\cdot}3.$$

The summation factor here is about 2·7 times Eddington's factor.

Next, consider calcium, of atomic number 20 and atomic weight 40. From Hartree's data,

$$\chi_{18}=1140,\quad \chi_{19}=5030,\quad \chi_{20}=5420,$$
$$\chi_{18}/kT=1{\cdot}39,\quad \chi_{19}/kT=6{\cdot}12,\quad \chi_{20}/kT=6{\cdot}60,$$

we find

$$\frac{x_{21}}{x_{20}}=e^{0{\cdot}90}=2{\cdot}46,\quad \frac{x_{20}}{x_{19}}=e^{1{\cdot}38}=3{\cdot}96,$$

whence

$$x_{21}=0{\cdot}66,\quad x_{20}=0{\cdot}27,\quad x_{19}=0{\cdot}07.$$

The summation factor is

$$2+\Sigma_{Ca}=2+4(0{\cdot}66\times6{\cdot}60+0{\cdot}27\times6{\cdot}12)+2\times0{\cdot}07\times1{\cdot}39=26{\cdot}24,$$

and the absorption coefficient is

$$k_{Ca}=0{\cdot}75\times26{\cdot}24\times\left(\frac{20}{26}\right)^2\times\frac{56}{40}=16{\cdot}3.$$

The summation factor here is about 3·1 times Eddington's factor.

This is about as favourable an element for a large absorption

coefficient as can be got from K-levels. We see that the ratio to the astronomical value has been improved from $\frac{1}{8}$ to $\frac{1}{3}$.

As an example of absorption governed by L-orbits, we take the case of silver. Hartree's values for the ionisation potentials of silver are

$\chi_{47}=30{,}000$	$\chi_{43}=6{,}500$	$\chi_{39}=5{,}500$
$\chi_{46}=29{,}000$	$\chi_{42}=6{,}200$	$\chi_{38}=5{,}300$
$\chi_{45}=6{,}980$	$\chi_{41}=6{,}000$	$\chi_{37}=2{,}300.$
$\chi_{44}=6{,}770$	$\chi_{40}=5{,}700$	

The mean of the 8 L-levels χ_{45} to χ_{38} is 6120 volts. This is so close to χ that we may apply case (3) above. The summation factor is then

$$2+\Sigma_{Ag}=2+2\times\tfrac{8}{9}\times 7{\cdot}50=15{\cdot}3$$

(1·8 times Eddington's), and the absorption coefficient (for atomic number 47 and atomic weight 108) is

$$k_{Ag}=0{\cdot}75\times 15{\cdot}3\times\left(\frac{47}{26}\right)^2\times\frac{56}{108}=19{\cdot}5.$$

Thus for silver the discrepancy ratio is about $\frac{2}{5}$.

It should be remembered that we have used Kramers' theoretical formula, which gives values smaller than the observed X-ray coefficients. He had to introduce a numerical factor of 2·2 or 2·4. Our modification of his procedure has partly discounted this, but even so, as shown in an earlier footnote (p. 754) of the present paper, we ought to multiply by a factor of about 1·3. This makes the values of the coefficients

$$k_{Fe}=8{\cdot}8,\quad k_{Ti}=18{\cdot}6,\quad k_{Ca}=21,\quad k_{Ag}=25,$$

and thus for, say, a mixture of calcium and silver the calculated value is half the astronomical value.

By a method somewhat different from his first method (more akin, in fact, to ours *), based on the experimental law of X-ray absorption, Eddington has calculated a number of special values for special elements. For iron, titanium, and silver he gives

$$k'_{Fe}=8{\cdot}4,\quad k'_{Ti}=10{\cdot}0,\quad k'_{Ag}=14{\cdot}3.$$

This agrees with our calculation of iron, whilst our values are nearly twice his for titanium and silver.

Finally we may follow Eddington in considering briefly the conditions at a point in Capella where the temperature is reduced to 1/3·6 of its central value. In virtue of this, k is increased by a factor of $(3{\cdot}6)^{\frac{1}{2}}=1{\cdot}9$, but the summation factor is altered also. From Eddington's calculation, $\chi/k\mathrm{T}$ becomes 9·42. For the new T, this makes $\chi=2160$ volts. According to Hartree, the eight L-potentials of the iron atom range from 2010 volts to 1150 volts. These are favourably placed with regard to χ, and the summation factor will amount to about $2+2\chi/k\mathrm{T}=20{\cdot}8$, instead of the value 8·95 at the centre. The absorption

* He associates an interval $4\nu_K-\frac{1}{2}\nu_K=\frac{7}{2}\nu_K$ with K-capture, $\frac{1}{2}\nu_K-\frac{1}{2}\nu_L=\frac{3}{2}\nu_L$ with L-capture. Our larger factors 4 and 2, instead of $\frac{7}{2}$ and $\frac{3}{2}$, can be seen to arise from captures in the higher levels when the K- (or L-) orbits are still vacant.

coefficient k_{Fe} is increased in the ratio 1 : 4·5 between the centre and the point considered. The value is now 30·5 (Eddington found 12·9), whilst the astronomical value (with Eddington's $\eta=1$) is 118·6. The discrepancy is now a factor of $\frac{1}{4}$. The empirical factor 1·3 improves k_{Fe} to 40, a discrepancy of $\frac{1}{3}$. For silver, the M-potentials range from 2300 volts downwards, at close intervals. These are fairly favourably situated, and assuming a Kramers' factor for M-levels of 1·5, the value of k_{Ag} at the new place is about 35·3, or 46, allowing for the empirical factor 1·3 ; this is about 0·4 of the astronomical value.

Eddington has pointed out that hydrogen gives results "differing in an extraordinary degree from every other substance." This arises, however, not so much from any intrinsic difference in the absorbing power of hydrogen nuclei, but from the fact that for a hydrogen star μ is 0·5, and therefore the calculated temperature, etc., are widely different. Nor do hydrogen and a heavy element mixed together give of themselves excessive absorption. In a mixture of a heavy element and hydrogen, enclosed in a given volume, the hydrogen is about twice as efficient as the heavy element in producing electrons to capture. Each hydrogen atom produces one electron, whilst the same mass of the heavy element produces roughly $(m_{Hyd}/m_{metal})\times(Z-2)$ times as many electrons, *i.e.* about half as many.

From this discussion I conclude that there is no very grave reason to suppose the theoretical value of the absorption coefficient to differ from the astronomical value more widely than by a factor of $\frac{1}{2}$. In fact the discussion might be taken to imply that the stars are not principally made up of iron (which for the conditions in Capella has unfavourably-placed ionisation potentials), but rather of some element like calcium. Of course in other stars other values would be obtained ; an increase in the central temperature from $9{\cdot}56\times10^6$ degrees to, say, 15×10^6 would bring the critical potential χ up to about 9650 volts, and would bring our summation factor up to about 22, giving k_{Fe} (apart from other trifling changes) equal to about 17. Whether this residual discrepancy of $\frac{1}{2}$ is significant it is difficult to say. Both the theoretical calculation and the astronomical one contain uncertainties. Here I should like to insist again on the arbitrary nature of the step in Kramers' argument which is concerned with the partition of the classical spectrum. A very small change in this in one particular direction would send up the final value considerably. It is not necessarily an adequate reply to this to say that Kramers' formula does in fact give the observed X-ray coefficients, because it is very doubtful whether it ought to do so. The fundamental step in his theory, the transition from absorption to emission *via* thermodynamic equilibrium, would appear to be invalid in calculating X-ray coefficients precisely, because the condition under which they are measured are not ones in which there is an equilibrium between the atoms and the X-radiation. The atoms have all their electrons still fastened to them, whilst the theory is really concerned with the K- and L-electrons as series electrons. Kramers' procedure, in fact, should be far more valid as a whole in the domain of stellar absorption coefficients than in that of X-ray coefficients ; the only question is as to whether great confidence may be placed in that essential part of his

procedure in which he divides up the classical spectrum by half quantum numbers.

7. *The Position of Eddington's Theory of Nuclear Capture.*—In this connection it may be recalled that Eddington has expressed "some hesitation in discarding" his theory of nuclear capture as an alternative to Kramers' theory. The theory of nuclear capture, according to Eddington's calculation, gave a value $k=78{\cdot}4$ instead of the astronomical value $47{\cdot}5$. But I think it is important to point out that when summation over all stages of ionisation is carried out, instead of the rougher process of averaging used by Eddington, it appears that here again a large dimensionless factor involving ionisation potentials appears.

By applying to formula (12), § 2*a*, the processes of integration and summation used in the present paper, I have shown * that the formula for the mean atomic absorption coefficient, on the theory of nuclear capture, is

$$a_c=\frac{15}{2\pi^3}\cdot\frac{Z(Z+2d/b)e^4h^3}{(2\pi m)^{\frac{3}{2}}}\cdot\frac{P_e}{(kT)^{\frac{9}{2}}}\cdot\left(1+\Sigma x_{r+1}\frac{\chi_r}{kT}\right) \quad . \quad (32)$$

whereas the formula given by Eddington † is

$$a_c'=\frac{15}{2\pi^3}\cdot\frac{Z(Z+2d/b)e^4h^3}{(2\pi m)^{\frac{3}{2}}}\cdot\frac{P_e}{(kT)^{\frac{9}{2}}}\times 2{\cdot}70 \quad . \quad . \quad (33)$$

The numerical factor $2{\cdot}70$ arises from Eddington's use of the mean quantum at temperature T, which is equal to $2{\cdot}70\,k$T, and is derived from the relation

$$\frac{h\bar{\nu}}{kT}=\int_0^\infty\frac{y^2dy}{e^y-1}\Big/\int_0^\infty\frac{y^3}{e^y-1}dy=2{\cdot}70.$$

In the paper mentioned I suggested that the summation factor in (32) would usually exceed $2{\cdot}70$. The data given in § 6 above allow us to evaluate this factor with ease, and we find :

Element.	Summation Factor.	k_c.
Fe	3·86	112
Ti	6·47	157
Ca	7·11	171
Ag	7·7	380

(The values of k_c have been calculated for the appropriate atomic numbers and atomic weights.) Comparison with the astronomical value $47{\cdot}5$ now shows the theory of nuclear capture to give values too large by factors ranging from $2{\cdot}4$ to 8 for these four elements. The ratio of the integrated coefficients a and a_c on the two theories is in fact, from (25) and (32),

$$\frac{a}{a_c}=\frac{32\pi}{3\sqrt{3}}\cdot\frac{e^2}{ch}\cdot\frac{2+\Sigma x_{r+1}(\chi_r/kT)(n_r-Q_r)^2/(n_r-\frac{1}{2}-Q_r)^2}{1+\Sigma x_{r+1}\chi_r/kT}$$
$$=2{\cdot}24\times10^{-2}\times\frac{2+\Sigma\ \ldots}{1+\Sigma\ \ldots},$$

* *Phil. Mag.*, **47**, 234, 1924, formulæ (30) and (32), with $\mu=\frac{1}{2}Z(Z+2d/b)e^4/m^2c^2$.
† *M.N.*, **83**, 38, 1923, equation (9).

which with a Kramers factor of 4 will be of the order of 0·09, with a factor of 2 of the order 0·05. The nuclear capture theory (as pointed out by Eddington) gives values of the order of 10 times those of Kramers' theory. The correcting factors obtained in this paper improve the position of Kramers' theory at the expense of the theory of nuclear capture.

8. *Summary.*—(1) The paper investigates afresh the theoretical value of the absorption coefficient of a highly-ionised gas, as based on Kramers' theory of X-ray coefficients. Account is taken of each atom in its appropriate state of excitation and stage of ionisation. The final formula, (25), differs from Professor Eddington's formula based on the same theory by a factor which arises from Kramers' application of the correspondence principle and involves also the actual ionisation potentials of the atoms concerned. This factor makes the absorption coefficient greater by a multiple which may amount to 3.

(2) The calculated values of the mass absorption coefficient at the centre of Capella are 6·8 for iron, 14·3 for titanium, 16·3 for calcium, and 19·5 for silver. When allowance is made for the fact that Kramers' theory gives the X-ray coefficients too small, the above values must be multiplied by about 1·3, making them 8·8, 18·6, 21, and 25. Professor Eddington's astronomical value is 47·5. The effect of the new factor disclosed in the present investigation is to reduce the discrepancy between the theoretical and astronomical values from the factor $\frac{1}{8}$ found by Professor Eddington to a factor lying between $\frac{1}{3}$ and $\frac{1}{2}$, if the star may be considered to be made up of calcium atoms or silver atoms. For iron, however, the discrepancy is still $\frac{1}{5}$; this is owing to the ionisation potentials for iron being unfavourably placed. Nearer the periphery of the star the discrepancy for iron improves to $\frac{1}{3}$, for silver becomes $\frac{2}{5}$.

(3) In view of the uncertainties in Kramers' theory, it is perhaps doubtful whether the outstanding discrepancy between theory and observation should now be considered significant. But all the absorption coefficients calculated in this paper are still subject to Rosseland's correction, and this will increase the discrepancy again.

(4) Professor Eddington's theory of nuclear capture, when corrected for the ionisation potentials concerned, leads to values of the absorption coefficient from 2·4 to 8 times the astronomical value.

Absorption Coefficients and the Pressure of Radiation in the Photospheric Layers of a Star. By Professor E. A. Milne.

§ 1. *Scope of the Paper.*—The purpose of this investigation is to apply to the outer layers of stars the formulæ for absorption coefficients obtained in the preceding paper. The main part of the investigation is concerned with the general opacity of the photospheric layers, but this is prefaced with a short evaluation of certain coefficients of selective absorption in the reversing layer.

In the preceding paper a formula has been found for the absorption coefficient of a gas, as a function of frequency, in the region of continuous absorption which extends on the high-frequency side of the limit of an optical series. This formula is applied in § 2 below to evaluate the absorption coefficient of hydrogen for the region extending beyond the Balmer limit $\lambda 3646$. It appears that the mass absorption coefficient near the Balmer limit, for atoms in the 2-quantum state, is of the order of 10^7, and that it should fall off towards the higher frequencies inversely as the cube of the frequency. The astrophysical importance of the corresponding band of absorption in certain celestial spectra has been pointed out by Wright * and Nicholson.†

As regards the question of the general opacity of the photospheric layers, the processes of capture and " partial capture " of electrons by ionised atoms, together with the reverse processes, must occur in the photospheric layers just as they do in the interior of a star. The only question is as to the extent to which they account for the total opacity. In the photospheric layers there are relatively much fewer free electrons than in the interior—the number will in fact be comparable with the number of atoms. This of itself would mean a much smaller absorption coefficient, since there are fewer electrons to capture and absorptions must balance captures. But the temperature being much lower, the electrons are travelling much slower, and are therefore more easily captured. This of itself would lead to a much larger value of the absorption coefficient.

In the preceding paper Eddington's formula for the absorption coefficient of an ionised gas has been re-derived by a method which takes strict account of every atom in its appropriate state of excitation and ionisation. The formula should therefore be just as applicable to an ionised gas at 6000° as it is at 6,000,000°, due attention being paid to the change of the relevant ionisation potentials. It states that the absorption coefficient κ is practically proportional to $P_e/T^{\frac{9}{2}}$, where P_e is the partial pressure of free electrons and T is the temperature. Now Eddington has shown that in the interior of a star P_e/T^4 is approximately constant (this is practically the constancy of the ratio of gas-pressure to radiation-pressure). If this law held right through to the photosphere, κ would in effect be proportional to $T^{-\frac{1}{2}}$, and since κ is of the order of 50 when $T=6\times10^6$, then κ would be of the order of 1500 when T is of the order of 6×10^3. This conjecture as to the order of magnitude of κ in the photospheric layers of a star turns out to be not far wrong for the sun, but it is an overestimate for giants.

It might be thought that an estimate of κ might be made by estimating P_e and T directly for some particular level in the photosphere and then applying the formula. An illustration of this mode of approach is given in § 3 below. In a temperature of 6000° and a pressure of 10^{-4} atmos., the general absorption coefficient comes out as $0{\cdot}8\times10^4$ for a gas consisting of calcium. But an estimate made in this way is quite unreliable. As we approach the surface of a star from the interior, the temperature tends to a limit whilst P_e tends to zero. It

* *Nature*, 1922, June 24, p. 810.
† *M.N.*, **85**, 253, 1925.

follows that κ tends to zero at the boundary. Thus to speak of "the value of κ" in a region in which κ is changing rapidly is meaningless without further precision. If, however, as might reasonably be conjectured, P_e/T^4 increases from 0 at the boundary to some approximately constant value in the immediate interior, then κ would behave approximately in the same way, and the limiting value which κ attained in the immediate interior could with justice be described as the value of κ in the photospheric layers. But the determination of the constant value of P_e/T^4 in the immediate interior is equivalent to the determination of the ratio of radiation-pressure to gas-pressure, and it is incorrect to suppose that in the immediate interior this ratio will have the value found by Eddington for the far interior. The ratio depends, in fact, on κ, the quantity we are wishing to determine. Thus we cannot apply our formula to determine κ without first considering the equilibrium of the photospheric layers. It is necessary to carry through for these layers the analysis Eddington has used for the far interior.

This is done in §§ 4, 5 below. The resulting values are $\kappa=1140$ for the sun, and $\kappa=133$ for the corresponding giant, taking calcium as a typical element.

These are not "astronomical values," in the sense in which Eddington has determined an astronomical value of the absorption coefficient in the far interior from the observed mass and luminosity of Capella. They are theoretical values, obtained from the theoretical formula for κ, but calculated for the values of P_e/T^4, which will ensure that the corresponding photospheres are in equilibrium with these values of κ. An observational check on the calculation arises, however, from the circumstance that when κ is known we can calculate the depth at which complete opacity sets in, and thus the value of the pressure at the base of the reversing layer. For the sun the pressure thus calculated comes out as $1{\cdot}1\times10^{-4}$ atmos., for the corresponding giant about $1{\cdot}3\times10^{-5}$ atmos. These are of the order of magnitude determined by the method of maximum intensities of absorption lines in stellar spectra, though a little lower than might have been anticipated. Further discussion is postponed until § 6.

§ 2. *The Absorption Coefficient of Hydrogen in the Region of Continuous Absorption beyond the Balmer Limit.*—It is convenient first to restate formula (10) of the preceding paper. Let e be the electronic charge, h Planck's constant, c the velocity of light. Suppose that an atom, of atomic number Z, possesses a sequence of optical levels $\chi^{(s)}$ given by a Rydberg formula

$$\chi^{(s)}=\chi^{(0)}(n-Q)^2/(n+s-Q)^2,$$

where $\chi^{(0)}$ is the lowest level, of total quantum number n, and Q is the quantum defect. When the series electron is occupying the $\chi^{(s)}$ level the atom can absorb radiation of frequency $\nu>\chi^{(s)}/h$ with ejection of the electron. Then if σ is the symmetry number * of the atom, $q^{(s)}$ the "weight" of the $\chi^{(s)}$ state, and q' the weight of the atom when ionised

* Number of equivalent outer electrons. For all levels except $s=0$, σ will be 1.

by the ejection of the electron, the *atomic absorption coefficient* for this process of absorption is

$$a^{(s)}(\nu)=\frac{16\pi^2}{3\sqrt{3}}\cdot\frac{Z^2e^6}{ch}\cdot\frac{q'\sigma}{q^{(s)}}\cdot\frac{\chi^{(s)}}{(h\nu)^3}(n+s-Q)^2\left[\frac{1}{(n+s-\tfrac{1}{2}-Q)^2}-\frac{1}{(n+s+\tfrac{1}{2}-Q)^2}\right] \quad (1)$$

The last two factors involving n, s, Q, reduce to

$$\frac{2(n+s-Q)^3}{[(n+s-Q)^2-\frac{1}{4}]^2},$$

which is approximately equal to

$$2/(n+s-Q).$$

The mass coefficient $\kappa^{(s)}(\nu)$ follows by dividing $a^{(s)}(\nu)$ by the mass of the atom.

We apply (1) to the Balmer limit of hydrogen. For the Balmer level, $n+s=2$, $Q=0$. Also $\sigma=1$, $q^{(1)}=6$, $q'=1$, $Z=1$. For the value of the constant in (1) we have

$$\frac{16\pi^2}{3\sqrt{3}}\cdot\frac{e^6}{ch}=1{\cdot}835\times10^{-39},$$

with

$$c=3\times10^{10},\quad h=6{\cdot}55\times10^{-27},\quad e=4{\cdot}774\times13^{-10},\quad e^6=1{\cdot}184\times10^{-56}.$$

This, divided by the mass of the hydrogen atom $1{\cdot}64\times10^{-24}$, gives $1{\cdot}120\times10^{-15}$. At the Balmer limit the frequency is $\nu^{(1)}$, where $h\nu^{(1)}=\chi^{(1)}$. The limit being $\lambda3646$, the frequency is $\nu^{(1)}=0{\cdot}822\times10^{15}$, and $h\nu^{(1)}=5{\cdot}38\times10^{-12}$. From these we find

$$a^{(1)}(\nu)=1{\cdot}20\times10^{-17},\quad \kappa^{(1)}(\nu)=0{\cdot}73\times10^{7}.$$

Each atom in the Balmer state is thus astonishingly opaque. It is easily calculated from this that a layer of atoms in the Balmer state at a temperature of 10,000° and a pressure of 1 atmos. would need to be only 0·1 cm. thick in order to cut down an incident beam to $1/e$ of its value; at a pressure of 10^{-4} atmos. a layer 10 metres thick would suffice. It has to be remembered, however, that in a mass of hydrogen at 10,000° only a small proportion of the atoms are actually in the 2-quantum (Balmer) state. It has been shown in a previous paper * that the maximum concentration of atoms in the Balmer state, for a partial electron pressure of $1{\cdot}3\times10^{-4}$ atmos., is a fraction $0{\cdot}77\times10^{-5}$. Thus averaged over all the atoms the mass coefficient is $0{\cdot}77\times10^{-5}\times0{\cdot}73\times10^{7}=56$. Even so a mass of hydrogen in this region is more opaque than the interior of a star ($\kappa=47$).

It is known that the average intensity of the absorption in the last few lines of the Balmer series shades off more or less continuously into the intensity of the absorption in the continuous region. This gives us reason to suppose that the mean mass coefficient of hydrogen in the Balmer lines is of the order of 10^7. I have suggested elsewhere † that monochromatic absorption coefficients are of this order of magnitude, or

* R. H. Fowler and E. A. Milne, *M.N.*, **84**, 509, 1924.

† *Phil. Mag.*, **47**, 210, 1924.

greater, and that absorption coefficients of this magnitude are consistent with the observed behaviour of the hydrogen lines in stellar spectra.* Formula (1) may be used similarly for other atoms, though there are no other interesting series-limits in the region of astrophysical importance. If we use (1) just to furnish an idea of the order of magnitude of the monochromatic coefficient for calcium, putting $Z=20$, atomic mass $=40\times1{\cdot}64\times10^{-24}$, $\sigma=1$, $q'/q=1$, and leaving $\chi^{(s)}/(h\nu)^3$ unaltered, we find

$$a(\nu)=2{\cdot}88\times10^{-14}, \quad \kappa(\nu)=4{\cdot}39\times10^{8}.$$

No precise meaning can be attached to the monochromatic absorption coefficient in a line unless the breadth (or conditions governing the breadth) is specified, but a value $\kappa(\nu)$ of the order of 4×10^8 is not inconsistent with that suggested by the astrophysical value of τ (the life of the excited atom) for Ca^+ atoms. For $d\lambda=0{\cdot}1$ Å, $\tau=0{\cdot}6\times10^{-8}$ sec. gives $\kappa(\nu)=8\times10^9$ for the H and K lines.† This larger value is to be expected; it corresponds to the concentration of the continuous absorption into a narrow range.

The value $\kappa(\nu)=1{\cdot}5\times10^7$ which we have found for the mass coefficient of hydrogen absorption beyond the Balmer limit could be used as a basis for a study of the intensity of the absorption in stellar spectra. It should be noticed from (1) that $\kappa(\nu)$ varies inversely as ν^3, and the absorption thus decreases as we move off towards the ultra-violet. This agrees qualitatively with what is observed. The actual absorption, *i.e.* the actual residual intensity, must depend, however, not only on $\kappa(\nu)$ but also on the nature of the equilibrium of the hydrogen atoms concerned, and this again will depend on "g" for the star, as it does for Ca^+ atoms in the solar chromosphere.

§ 3. *Crude Estimate of the Order of Magnitude of the General Coefficient of Absorption in the Photosphere of a Star.*—As above, we begin by restating the formula for the general coefficient of absorption found in the previous paper. Of the atoms of an ionised gas, let there be a fraction x_1 which are neutral and in their lowest state, and let their ionisation potential be χ_1; let there be a fraction x_2 which are once ionised and in their lowest state, and let their ionisation potential be χ_2; and so on. The symbols n_r, Q_r are to denote the total quantum number and the quantum defect of the x_r-atoms. The mass of the atom is m', the mass of the electron is m, and k denotes Boltzmann's gas-constant. Then at temperature T and partial electron pressure P_e, the mass absorption coefficient κ and the atomic absorption coefficient a for integrated radiation are given by ‡

$$m'\kappa=a=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6h^2}{c(2\pi m)^{\frac{3}{2}}}\cdot\frac{P_e}{(kT)^{\frac{7}{2}}}f \quad . \quad . \quad . \quad (2)$$

where

$$f=1+\sum_{r=1}x_{r+1}\left(1+\frac{\chi_r}{kT}\cdot\frac{(n_r-Q_r)^2}{(n_r-\tfrac{1}{2}-Q_r)^2}\right).$$

* *Loc. cit.*, footnote, p. 227.

† "An astrophysical determination of the average life of an excited calcium atom," *M.N.*, **84**, 358, 1924.

‡ Formulæ (3) and (24) of the preceding paper.

Purely for the sake of ascertaining the order of magnitude of this expression under some given conditions, we shall evaluate it for calcium under the conditions

$$T=6000^\circ, \quad P_e=10^{-4} \text{ atmos.}=10^2 \text{ dyne cm.}^{-2}.$$

Under these circumstances, it is readily found that all the calcium atoms are once ionised. Thus $x_1=0$, $x_2=1$, $x_3=\ldots=0$. At this temperature kT is 0·516 volts, whilst χ_1 for Ca is 6·08 volts. From the series limit (1S-term) we find that $n-Q$ is about 1·5, and hence f comes to 28·5. On using

$$m=9\cdot00\times10^{-28}, \quad (2\pi m)^{\frac{3}{2}}=4\cdot25\times10^{-40}, \quad k=1\cdot37\times10^{-16},$$

we find that

$$\frac{80}{\pi^2\sqrt{3}}\cdot\frac{e^6h^2}{c(2\pi m)^{\frac{3}{2}}}=1\cdot87\times10^{-79}, \quad (kT)^{\frac{3}{2}}=4\cdot14\times10^{-55},$$

and hence putting $Z=20$, and the mass of the calcium atom equal to $40\times1\cdot64\times10^{-24}$, we find from (2)

$$a=5\cdot15\times10^{-19}, \quad \kappa=0\cdot78\times10^4.$$

For other atoms, apart from complications arising from the ionisation potentials, the value is proportional to Z^2/W, where W is the atomic weight.

According to this estimate the opacity of the photospheric layers is over 100 times that of the interior. We shall see later that in fact it is an overestimate; the theory will require a smaller value of $P_e/(kT)^{\frac{3}{2}}$. But there is nothing absurd in so big an absorption coefficient from the point of view of the equilibrium of the photospheric layers. If T_1 is the effective temperature of a star, the flux of radiation leaving it is σT_1^4, where σ is Stefan's constant, $5\cdot70\times10^{-5}$. The upward acceleration due to the pressure of radiation, if the absorption coefficient is κ, is $\kappa\sigma T_1^4/c$, and a state of equilibrium will be possible provided

$$\frac{\kappa\sigma T_1^4}{c}<g,$$

i.e. provided

$$\kappa<\frac{cg}{\sigma T_1^4} \qquad . \quad . \quad . \quad . \quad (3)$$

For $T_1=6000^\circ$, $g=2\cdot74\times10^4$ (solar conditions), the right-hand side of (3) comes to $1\cdot114\times10^4$, and so a value $\kappa=0\cdot78\times10^4$ satisfies (3). It does not preclude equilibrium, though it would involve a radiation-pressure equal to 0·7 of gravity. We see that dynamically there is nothing impossible in a photospheric absorption coefficient of the order of 10^4 for the sun. For a giant star, in which g is much smaller, the maximum permissible value of κ is much smaller. The lower values of P_e in the atmosphere of a giant will, however, ensure that κ is in fact smaller.

§ 4. *The Absorption Coefficient and the Pressure of Radiation in the Immediate Interior of a Star.*—We shall prove in § 9 that as we penetrate the photospheric layers from outside we first encounter a layer in which

the ratio of gas-pressure to radiation-pressure increases rapidly from zero, thereafter a region in which the ratio alters very slowly. This latter region we shall term " the immediate interior " of the star. The ratio of gas-pressure to radiation-pressure in the immediate interior changes only with the change in $T^{\frac{1}{2}}$, and it gradually passes into the value calculated by Eddington for the " far interior." For our purpose, this slowly varying factor $T^{\frac{1}{2}}$ (slowly varying, that is, compared with the pressure of radiation, which varies as T^4) may be replaced by some mean and treated as a constant. Thus what we shall prove in § 9 is that in the immediate interior the ratio of gas-pressure to radiation-pressure is approximately constant.

It will be convenient at this stage to assume the result just stated and examine the consequences.

Let p be the total pressure, p' the pressure of radiation. Our assumption is that in the immediate interior

$$\frac{p}{p'}=\text{const.}=\lambda_1=\frac{\beta}{1-\beta} \quad . \quad . \quad . \quad . \quad (4)$$

say, where λ_1 and β are constants. Since p and p' are proportional, so are their gradients, and hence they are in the same ratio as the accelerations to which they give rise. Now radiation-pressure produces an acceleration $\kappa\sigma T_1^4/c$, and the combined accelerations are equal and opposite to an acceleration g. Hence

$$\frac{p+p'}{p'}=\frac{g}{(\kappa\sigma T_1^4)/c}=\frac{cg}{\kappa\sigma T_1^4} \quad . \quad . \quad . \quad (5)$$

For brevity put

$$A=\frac{80}{\pi^2\sqrt{3}}\cdot\frac{Z^2e^6h^2}{c(2\pi m)^{\frac{3}{2}}m'},$$

where m' is the mass of the atom. Then

$$\kappa=Af\frac{P_e}{(kT)^{\frac{5}{2}}}.$$

Now put

$$p'=\tfrac{1}{3}aT^4=\tfrac{1}{3}a'(kT)^4.$$

It is easily verified that the value of a' is $8\pi^5/15c^3h^3$, which is $2{\cdot}16\times10^{49}$. Let us now suppose that the partial electron pressure P_e is some fraction θ of the total pressure p. We have then

$$\kappa=\frac{\frac{1}{3}a'A\theta f}{(kT)^{\frac{1}{2}}}\cdot\frac{p}{p'}=\xi\frac{p}{p'} \quad . \quad . \quad . \quad . \quad (6)$$

say. The coefficient ξ is not a constant, but we are going to treat it as such, taking a mean value for $T^{\frac{1}{2}}$. The fraction θ will increase from a value near $\frac{1}{2}$ up to nearly unity in the far interior; the summation factor f will vary irregularly, probably on the whole decreasing.

Eliminating κ between (5) and (6) and putting λ_1 for p/p' we find

$$\lambda_1(\lambda_1+1)=\frac{\beta}{(1-\beta)^2}=\frac{1}{\xi}\cdot\frac{cg}{\sigma T_1^4} \quad . \quad . \quad . \quad (7)$$

The ratio λ_1 of gas-pressure to radiation-pressure is thus determined as the positive root of a certain quadratic equation.

Equation (7) bears a close resemblance to Eddington's formula for the luminosity of a star.* If L, M, r_0 are the luminosity, mass, and radius of the star, then

$$L=4\pi r_0^2\sigma T_1^4, \quad g=GM/r_0^2,$$

whence

$$\frac{cg}{\sigma T_1^4}=\frac{4\pi cGM}{L} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence (7) may be written

$$L=\frac{4\pi cGM}{\xi}\cdot\frac{(1-\beta)^2}{\beta} \quad . \quad . \quad . \quad . \quad (9)$$

Eddington's formula is

$$L=\frac{4\pi cGM}{(\xi\eta)'}\cdot\frac{(1-\beta')^2}{\beta'} \quad . \quad . \quad . \quad (10)$$

where the prime (′) denotes that the corresponding quantities are to be evaluated for the far interior. The quantity η is a numerical factor equal to unity in the "outer parts" of the far interior.

The quantity β' is determined from Eddington's quartic equation; it depends only on the mass of the star and the mean molecular weight in the far interior. But for the quantity β we have no independent determination of this character; it must be regarded as determined by (9) from the observed value of L/M (or $\sigma T_1^4/g$) and the value of ξ given by the theory of the absorption coefficient.

In applying (10) to deduce an astronomical value of ξ (*i.e.* of κ), Eddington estimates η for the centre of the star—an estimate which is rather uncertain. It may be asked, why does he not rather take a point in the "immediate interior" where η is known to be unity? Equation (9) provides the answer. It is perfectly legitimate to apply (10) to the outer parts of the star, only β here is not known. The quartic equation, in a way, gives the average value of β for the star as a whole, and throughout the greater part of the star β is so constant that its actual value there is equal to its mean value. But near the periphery the actual value departs considerably from the mean value. In discussing the photospheric layers we are apparently one equation short as compared with the far interior, but this is inevitable. We cannot deduce for a particular portion of a star as much as we can deduce for the star as a whole, from dynamical principles only. Eddington, too, is one equation short when considering any one particular point in the star, namely the value of η. The latter deficiency will be made up only when we know how the energy is generated—a problem in nuclear physics. In considering the photosphere we are able to supply the deficiency more easily; we require to know simply the physics of atomic absorption—the constant ξ. The star adjusts its $1-\beta$ for its outer layers to suit the physical constant ξ for the atoms present. If at any moment $1-\beta$ is too large for equilibrium, the star's outer parts will expand, causing a reduction of pressure, hence a decrease in κ and so a

* "The Masses and Luminosities of the Stars," *M.N.*, **84**, 328, 1924.

decrease in radiation-pressure, until equilibrium is restored. Equation (7), and equation (11) below, are simply the mathematical expression of this tendency.

Putting $\eta=1$ in (10), we have on comparison

$$\frac{(1-\beta)^2}{\beta\xi}=\frac{(1-\beta')^2}{\beta'\xi'}.$$

Now $\xi \propto T^{-\frac{1}{2}}$; ξ for the immediate interior corresponds to a temperature of the order of 10^4, ξ' for the outer parts of the far interior to a temperature of the order of 10^6. Thus ξ is considerably bigger than ξ'. It follows that $(1-\beta)^2/\beta$ is considerably bigger than $(1-\beta')^2/\beta'$. Thus radiation-pressure is more important in the immediate interior than in the far interior. We shall estimate this effect numerically in § 5 below.

When (7) has been solved, κ follows at once from $\kappa=\lambda_1\xi$. But we can write down an equation for κ directly.

Eliminating p/p' between (5) and (6) we have

$$\frac{\kappa^2}{\xi}+\kappa-\frac{cg}{\sigma T_1{}^4}=0 \quad . \quad . \quad . \quad . \quad (11)$$

The absorption coefficient κ in the immediate interior of any star is thus determined as the positive root of a certain quadratic. It will be seen that κ as determined by (11) always automatically satisfies inequality (3).

§ 5. *Numerical Values.*—For the sun, with the data already given, we find for calcium $A=1\cdot140\times10^{-54}$. For definiteness we shall take $\theta=\frac{1}{2}$; this is exactly true for a mass of once-ionised atoms. We had also $f=28\cdot5$. Taking the residual T in ξ as 6000°, we find $\xi=1\cdot293\times10^2$. We found previously $cg/\sigma T_1{}^4=1\cdot114\times10^4$. Hence the equation for λ_1 is

$$\lambda_1(\lambda_1+1)=0\cdot862\times10^2.$$

Solving, we find

$$\lambda_1=8\cdot80, \quad 1-\beta=(1+\lambda_1)^{-1}=0\cdot102, \quad \kappa=1140.$$

For Capella, Eddington has adopted a mass $4\cdot2\odot$, and a mean density ρ_m of 0·0026. Since the sun's density is 1·41, and $g \propto M^{\frac{1}{3}}\rho_m{}^{\frac{2}{3}}$, we find that g for Capella is 0·0243 times the solar value. Capella actually has an effective temperature some 500° less than the sun, but to simplify our arithmetic we shall take it as having the same temperature, 6000°. We thus compare the photospheres of two stars of the same effective temperature, in one of which g is 41 times that in the other. The value of $cg/\sigma T_1{}^4$ is now $2\cdot46\times10^2$. The value of ξ is unaltered. Hence for Capella

$$\lambda_1(\lambda_1+1)=2\cdot10,$$

which gives

$$\lambda_1=1\cdot033, \quad 1-\beta=0\cdot493, \quad \kappa=133\cdot5.$$

Thus the mass absorption coefficients in the photospheres of the sun and of Capella, as derived from Kramers' formula, are $\kappa=1140$ and $\kappa=133\cdot5$, taking calcium as a typical element. They are thus about 20 times and 3 times Eddington's value for the far interior respectively. Any empirical correction to Kramers' formula which has the effect of reducing the discrepancy between the theoretical and astronomical values

of κ for the far interior will imply even larger absorption coefficients for the photosphere than we have found. As regards the effect of taking atoms other than calcium, the chief difference is that ξ is proportional to Z^2/W, where Z is the atomic number and W is the atomic weight. For calcium Z^2/W is 10; for hydrogen it is 1; for iron it is 12; and for silver it is 20·5. When g is large, the photospheric value of κ varies as $\xi^{\frac{1}{2}}$; when g is small, it varies less rapidly than $\xi^{\frac{1}{2}}$. It is clear that save for hydrogen * the order of magnitude of κ is the same for all elements.

The paradoxical result should be noted that in the dwarf stars the value of κ is the greater, but radiation-pressure is the less important. The increase of κ due to increase of pressure in the photosphere of the dwarf stars is far from neutralising the increase in g.

Eddington's values of $1-\beta$ for the far interiors of the sun and of Capella are 0·050 and 0·283. These should be compared with the values found above for the immediate interiors, 0·102 and 0·493. As anticipated in § 4, radiation-pressure is more important in the immediate interior than in the far interior. In the sun it is almost twice as important; in Capella about 1·7 times. In the photosphere of the sun, radiation-pressure is $\frac{1}{10}$ of the total pressure; in the interior of the sun, it is $\frac{1}{20}$. The fractions for Capella are $\frac{1}{2}$ and $\frac{3}{10}$.

§ 6. *Pressures.*—I have shown in an earlier paper † that an approximate solution of the integral equation for radiative equilibrium, valid near the boundary, is

$$T^4=\tfrac{1}{2}T_1{}^4(1+\tfrac{3}{2}\tau) \quad . \quad . \quad . \quad . \quad (12)$$

where τ is the optical depth measured inwards,

$$\tau=\int_0^x \kappa\rho dx,$$

and T_1 is the effective temperature, $\sigma T_1{}^4$ being the net flux per unit area. This solution gives the correct net flux both for τ large and for $\tau=0$.

The latter statements are in fact easily verified. For the first, we have that at depth τ the outward intensity $I(\tau, \theta)$ in a direction making θ with the axis of x, $(0\leqslant\theta<\frac{1}{2}\pi)$ is

$$I(\tau, \theta)=\int_0^\infty (\sigma/\pi)T^4 e^{-(t-\tau)\sec\theta}\sec\theta\, dt$$

$$=\frac{\sigma T_1{}^4}{2\pi}(1+\tfrac{3}{2}\tau+\tfrac{3}{2}\cos\theta) \quad . \quad . \quad . \quad . \quad (13)$$

For values of τ so large that the effect of the boundary may be neglected, (13) also gives the intensity for $\frac{1}{2}\pi<\theta\leqslant\pi$. Hence for τ large the net outward flux is

$$2\pi\int_0^\pi I(\tau, \theta)\cos\theta\sin\theta d\theta=\sigma T_1{}^4,$$

* For hydrogen, f will be about 1 under solar conditions, instead of 28 for Ca. We then find $\lambda_1=156$, and κ is about 72. It appears that hydrogen contributes but little to the general opacity under solar conditions.

† *M.N.*, **81**, 361, 1921.

as stated. It may be noted that on differentiating (12) we have

$$\frac{dT^4}{dx}=\tfrac{3}{4}T_1{}^4\kappa\rho,$$

and on inserting $p'=\frac{1}{3}aT^4$, $a=4\sigma/c$, this becomes

$$\frac{c}{\kappa\rho}\frac{dp'}{dx}=\sigma T_1{}^4 \quad . \quad . \quad . \quad . \quad (14)$$

This is the well-known relation connecting the net flux with the gradient of radiation-pressure, which on approximation (12) is thus valid everywhere and not merely in the interior. (We make use of this in § 9.)

For the second, by (13) the emergent intensity at the boundary is

$$I(0,\theta)=\frac{\sigma T_1{}^4}{2\pi}(1+\tfrac{3}{2}\cos\theta) \quad . \quad . \quad . \quad (15)$$

and hence the total emergent radiation is

$$2\pi\int_0^{\frac{1}{2}\pi} I(0,\theta)\cos\theta\sin\theta d\theta=\sigma T_1{}^4,$$

which again is the correct value.* Equation (15) implies a coefficient of darkening of the sun's disc towards the limb of $\frac{3}{5}$, which is very close to the observed value. Given the law of darkening of the disc from observation, the distribution of temperature in the immediate interior giving rise to it is determinate,† and thus the observed darkening of the sun's disc is direct evidence that (12) gives the actual temperature distribution at the boundary and in the immediate interior.

Multiplying (12) by $\frac{1}{3}a$, we have

$$p'=\tfrac{1}{6}aT_1{}^4(1+\tfrac{3}{2}\tau) \quad . \quad . \quad . \quad . \quad (16)$$

and hence in the immediate interior

$$p=\lambda_1 p'=\tfrac{1}{6}\lambda_1 aT^4(1+\tfrac{3}{2}\tau) \quad . \quad . \quad . \quad (17)$$

Light originating at a depth τ is cut down by fraction $e^{-\tau}$ in the course of its transit through the overlying matter. If $\tau=4{\cdot}5$, $e^{-\tau}=0{\cdot}011$, and only 1 per cent. emerges. Roughly speaking, we may say that complete opacity is established at a depth $\tau=4{\cdot}5$; further than this into the photosphere we cannot see, and we can consider this depth to mark the base of the reversing layer. At the depth $\tau=4{\cdot}5$, by (17) the gas-pressure $p_{\text{phot.}}$ has the value

$$p_{\text{phot.}}=\tfrac{3}{2}\tfrac{1}{4}\lambda_1 aT_1{}^4 \quad . \quad . \quad . \quad . \quad (18)$$

This is the formula for the pressure in a stellar atmosphere at the bottom of the reversing layer, λ_1 being given by (7).

For $T_1=6000°$, $aT_1{}^4$ is 9·85. Hence for the sun, ($\lambda_1=8{\cdot}80$), we have

$$p_{\text{phot.}}=1{\cdot}12\times10^2 \text{ dynes cm.}^{-2}=1{\cdot}12\times10^{-4} \text{ atmos.}$$

* The approximation recently used by Jeans, $T^4=\frac{3}{8}T_1{}^4(1+2\tau)$ gives the correct net flux $\sigma T_1{}^4$ for τ large and satisfies (14), but gives the net flux at the boundary as $\frac{2}{3}\sigma T_1{}^4$ instead of $\sigma T_1{}^4$ (*M.N.*, **85**, 198, 1925).

† Milne, *Phil. Trans. Roy. Soc.*, **223** A, 214, 1922. Lundblad, *Astrophys. Journ.*, **58**, 113, 1923.

For Capella, ($\lambda_1 = 1{\cdot}033$), we have

$$p_{\text{phot.}} = 1{\cdot}32 \times 10 \text{ dynes cm.}^{-2} = 1{\cdot}32 \times 10^{-5} \text{ atmos.}$$

These are in general agreement as regards order of magnitude, with estimates of $p_{\text{phot.}}$ made in other ways. Russell and Stewart * have concluded from a variety of considerations that the pressure of the photospheric gases is less than 10^{-2} atmos., and that the average pressure in the reversing layer is not greater than 10^{-4} atmos., though I am unable to agree with all the details of their lines of argument. The observed occurrences of maxima of absorption lines in the stellar sequence of spectra have been shown to be in agreement with the predictions of the theory of high-temperature ionisation provided the partial electron pressure P_e in the layer producing a strong line of a subordinate series is about $1{\cdot}3 \times 10^{-4}$ atmos.; in the region in which strong principal lines originate the pressure P_e is about 10^{-7} atmos.† These are averages over numbers of stars; they depend on the correctness of the existing determination of the colour-temperature of the star; and they ignore the differences between giants and dwarfs. We should have expected $p_{\text{phot.}}$ as defined above to come out slightly higher than P_e for the reversing layer. It is striking that the order of magnitude, 10^{-4} atmos., obtained by methods having nothing in common should be the same, but actually $p_{\text{phot.}}$ on our calculation is somewhat lower than the spectral determinations of P_e for reversing layers. But the P_e's determined by Fowler and Milne rested on the assumption that the colour-temperature of the star was the actual temperature of the reversing layer. The reversing layer is likely to be somewhat cooler, perhaps at the theoretical boundary temperature $T_0 = T_1 \times 2^{-\frac{1}{4}} = T_1 \times 0{\cdot}84$, and this would necessitate lower values of P_e to fit the observed maxima.

The determination of P_e from spectra is in effect a fairly direct determination from observation; we use our observations of the effective temperatures at which particular absorption lines reach maxima. On the other hand, the foregoing method of finding $p_{\text{phot.}}$ is based entirely on atomic theory and involves no empirical constants. It is interesting to examine upon exactly what assumptions our simple formula (18) depends. We have seen that (16) is directly implied by the observed darkening of the solar disc towards the limb. Thus atomic theory only comes in in the determination of the constant λ_1 or $\beta/(1-\beta)$, from equation (7).

To obtain a value of $p_{\text{phot.}}$ tenfold higher, we should need a value of ξ 100-fold lower. This would imply absorption coefficients near a series limit to be 100 times smaller than the values found in § 2. There are no direct determinations of the absorption coefficients in the region beyond a series limit, but smaller values than those calculated would be difficult to reconcile with the observed values of monochromatic absorption coefficients, such as those of Wood for λ2537 Hg and the

* *Astrophys. Journ.*, **59**, 197, 1924. One of their methods was to extend a formula of Eddington's as it stands to the outer region of the sun. This gives, as it happens, the right order of magnitude, but they used Eddington's value of $(1-\beta)$ for the far interior, which we have seen is too small.

† R. H. Fowler and E. A. Milne, *M.N.*, **83**, 403, 1923; **84**, 499, 1924.

experimental determinations of Fuchtbauer and others. Formula (2) is derived from Kramers' formula for X-ray absorption coefficients, which gives values in accord with experiment. Applied to the far interior of a star, it gives values rather smaller than the observed astronomical value. In applying it to the optical region and the immediate interior of a star, we are in effect extrapolating from the X-ray region into the optical region, but the method of derivation of (2) is independent of whether X-ray frequencies or optical frequencies are involved. Perhaps for no element is the mode of derivation of (2) more strictly valid than for hydrogen, for which the value of ξ is some 280 times smaller than for calcium.* For hydrogen, λ_1 is 156, which is 17·7 times bigger than for calcium, and so $p_{\text{phot.}}$ for hydrogen is about 2×10^{-3} atmos. This, then, is an upper limit to the photospheric pressure.

The only reasonable doubt that can be entertained about the extension of Kramers' work to the optical electrons of elements of fairly high atomic number is as to whether the factor Z^2 should not be corrected to some extent for the screening of the nucleus by the bound electrons. The method assumes that the radiation from the electron during the encounter is the same as if it were encountering a bare nucleus. However improbable this may seem, the assumption did in fact work for X-rays. But if we go to extremes and replaced Z^2 by unity, we should only reduce ξ for Ca by a factor of 400, thereby increasing λ_1 and $p_{\text{phot.}}$ by a factor of 20. Thus it seems impossible, even with gross tampering with Kramers' formula, to obtain values of $p_{\text{phot.}}$ higher than 10^{-3} atmos. Until a better formula for κ is available I prefer to accept Kramers' formula as it is, and with it the calcium values $p_{\text{phot.}}=1\cdot12\times10^{-4}$ atmos. for the sun and $p_{\text{phot.}}=1\cdot32\times10^{-5}$ atmos. for Capella.

§ 7. *Comparison with Russell and Stewart's Calculations.*—A direct estimate of the opacity of a stellar photosphere from photo-electric emissions and absorptions on the lines of the present paper does not seem to have hitherto been attempted. Russell and Stewart † have calculated the opacity as arising from absorption by free electrons, from scattering by free electrons and from scattering by bound electrons. It will now be shown that the opacities calculated in the present paper are enormously higher than those calculated by Russell and Stewart. From Russell and Stewart's Table I., a column of partially ionised sodium vapour at a temperature of 5000° and a pressure of 10^{-4} atmos. must have a length of 200,000 kms. to cut down a beam of light to $(1/e)$th of its value.

We shall repeat the calculation with our own value of the absorption coefficient, for calcium at 6000°. If l is the length of column, of density ρ, required to cut down the beam to $(1/e)$th of its value, then

$$\kappa\rho l=1 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

But

$$\kappa=\xi\frac{p}{\frac{1}{3}aT^4}, \qquad \frac{p(1-\theta)}{\rho}=\frac{k}{m'}T.$$

* We are assuming $\theta=\frac{1}{2}$ as before, *i.e.* assuming there are plenty of free electrons provided by more easily ionisable elements.

† *Loc. cit.*

Hence from (19),

$$l=\frac{\frac{1}{3}aT^4}{\xi p^2(1-\theta)}\cdot\frac{kT}{m'}.$$

Inserting the values previously used, this gives

$$l=\frac{6\cdot4\times10^{10}}{p^2} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

For $p=10^{-4}$ atmos. $=10^2$ dynes cm.$^{-2}$, this gives $l=6\cdot4\times10^6$ cm. $=64$ kms. For 5000° this would want decreasing in the ratio $(5/6)^{\frac{3}{2}}$ or 0·43, giving 28 kms. Thus according to our work a vapour such as calcium or sodium is about 7000 times as opaque, at 10^{-4} atmos., as calculated by Russell and Stewart. This simply means that photo-electric opacity is dominant; the sources of opacity enumerated by Russell and Stewart, if our work is correct, are negligible in comparison.

That the opacity determined by Russell and Stewart is an underestimate is rendered probable by the fact that it only enabled them to deduce that the pressure at the base of the chromosphere is less than 10^{-4} atmos. This being the pressure for *subordinate* lines in the reversing layer, and the reversing layer pressure for principal lines being 10^{-7} atmos., the actual pressure at the base of the chromosphere must be much smaller. We should anticipate it to be definitely less than 10^{-7} atmos. Our calculations permit an immediate estimate. For, as Russell and Stewart mention, just outside the apparent limb of the sun, *i.e.* at the base of the chromosphere, the line of sight runs for a distance of the order of 10^5 kms. without appreciable absorption. If we take "without appreciable absorption" to mean an intensity less than 1 per cent. of the corresponding black-body intensity, then τ for a column 10^5 km. long must be $<0\cdot01$, and hence applying (19) we have

$$\kappa\rho\times10^{10}<0\cdot01,$$

or, by the same arguments that lead to (20),

$$p^2_{\text{chrom.}}<\frac{6\cdot4\times10^{10}\times0\cdot01}{10^{10}}\times\left(\tfrac{5}{6}\right)^{\frac{3}{2}}=2\cdot5\times10^{-2},$$

whence

$$p_{\text{chrom.}}<1\cdot6\times10^{-1}\text{ dynes cm.}^{-2}=1\cdot6\times10^{-7}\text{ atmos.}$$

This is in agreement with the above conjecture. If anything, we should have expected a smaller pressure, so that in this connection our formula appears if anything to underestimate the opacity rather than overestimate it.

It should be noted that the huge general opacity we have found is a complete answer to the old difficulty as to how a gaseous sun can have a sharp edge, as also to the similar difficulty as to how a gas can give a continuous spectrum. The opacity arises almost entirely from the presence of free electrons. In their encounters with atoms they are captured or partially captured, with emission of radiation, and the converse processes give rise to absorption. Put another way, the opacity arises from the easy detachability of the series-electrons of

atoms. At the temperature of a stellar atmosphere, the atoms cannot help being ionised, either from their normal states or from higher excited states. The radiative loss of an electron is accompanied with absorption; and the electron thus free may again contribute to the total opacity by absorbing during an encounter with another atom. Our argument is, moreover, entirely independent of the extent to which atoms are ionised by processes other than radiative ones. Kramers' method amounts to a determination of the actual number of radiative ionisations per second.

The rapidity of the transition from an opaque photosphere to a transparent chromosphere is helped by the circumstance that by (20) the opacity of a column of vapour of given length decreases as the square of the pressure.

One further point may be mentioned in illustration of the physical meaning of equation (11). For given P_e and T, κ varies as ξ, but equation (11) shows that in a stellar atmosphere κ varies as $\xi^{\frac{1}{2}}$ approximately. The reason is easy to see. If ξ is increased, the opacity of the photospheric layers is increased, and hence the effect of radiation-pressure is increased. The resulting expansion causes a general reduction of pressure in the photospheric layers, which gives rise to a reduction in κ which is superimposed on the increase due to ξ. The resultant increase is proportional only to $\xi^{\frac{1}{2}}$.

§ 8. *Giants and Dwarfs.*—Solving (7) for λ_1 and inserting in (18), we have

$$p_{\text{phot.}}=\tfrac{3}{2}\tfrac{1}{4}a\mathrm{T}_1{}^4\times\tfrac{1}{2}[-1+(1+4cg/\sigma\mathrm{T}_1{}^4\xi)^{\frac{1}{2}}] \quad . \quad . \quad (21)$$

We can deduce from this the ratio of the pressures in the photospheres of a giant and a dwarf at the same effective temperature. If p_1, g_1 refer to the giant, p_2, g_2 to the dwarf, then

$$\left(\frac{p_1}{p_2}\right)_{\text{phot.}}=\frac{-1+(1+4cg_1/\sigma\mathrm{T}_1{}^4\xi)^{\frac{1}{2}}}{-1+(1+4cg_2/\sigma\mathrm{T}_1{}^4\xi)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (22)$$

Consideration of the numerical values occurring in practice shows that in most cases this is approximately equivalent to

$$\left(\frac{p_1}{p_2}\right)_{\text{phot.}}=\left(\frac{g_1}{g_2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (23)$$

(For the sun and Capella, $(g_2/g_1)^{\frac{1}{2}}$ is 6·42, whilst the values we have obtained for $p_{\text{phot.}}$ give $p_2/p_1=8{\cdot}48$.)

Relation (23) replaces the well-known crude result (based on a theory of constant κ) that p_1/p_2 is equal to g_1/g_2. The law of increase of κ with P_e prevents the change of gravity from a giant to a dwarf from having its full effect on the pressure.

There is some observational evidence that the change of pressure between a giant and a dwarf is less severe than $p_1/p_2=g_1/g_2$ and more like that given by (23). It is well recognised that classified according to line spectra giants have lower effective temperatures than dwarfs of the same type. "Same type" means in effect "same ionisation,"

and hence we write as the equation connecting the temperatures T_1 and T_2 of a giant and dwarf of the same type,

$$\frac{T_1^{\frac{5}{2}}e^{-\chi/kT_1}}{p_1}=\frac{T_2^{\frac{5}{2}}e^{-\chi/kT_2}}{p_2} \quad . \quad . \quad . \quad (24)$$

where χ is the ionisation potential for the spectra used to define the type, *i.e.* in practice the spectra having a maximum in the type in question. If we apply this * to the sun and Capella, with $p_2/p_1=g_2/g_1=40{\cdot}7$, putting $\chi=8$ volts, $T_2=6000°$, we find $T_1=4950°$. A difference $T_2-T_1=1050°$ is, however, somewhat larger than is observed.

But if we employ in (24) our ratio $p_2/p_1=8{\cdot}48$, we find $T_1=5330$. Actually, however, as we are now taking account of the differing temperatures of the giant and the dwarf, $p_{\text{phot.}}$ should first be calculated from (21) with an estimated value of T_1, assuming $T_2=6000°$ as before. This slightly alters p_2/p_1 from 8·48 to 9·80, and we then find $T_1=5300°$. Probably it would be more accurate to replace T_1 and T_2 in (24) by the actual reversing layer temperatures, say $2^{-\frac{1}{4}}T_1$ and $2^{-\frac{1}{4}}T_2$. This makes $T_1=5400°$, $T_2-T_1=600°$. The value tabulated by Seares derived from Mount Wilson colour-indices is $T_2-T_1=470°$ for type Go.

For two stars of different T as well as different g, (21) leads to the approximation

$$\left(\frac{p_2}{p_1}\right)_{\text{phot.}}=\left(\frac{T_2}{T_1}\right)^2\left(\frac{g_2}{g_1}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (25)$$

which gives for the connection between the effective temperatures of a giant and dwarf of the same type

$$\frac{T_1^{\frac{1}{2}}e^{-\chi/kT_1}}{g_1^{\frac{1}{2}}}=\frac{T_2^{\frac{1}{2}}e^{-\chi/kT_2}}{g_2^{\frac{1}{2}}} \quad . \quad . \quad . \quad (26)$$

This gives

$$\frac{T_2}{T_1}-1=\frac{\log(g_2/g_1)-\log(T_2/T_1)}{2(\chi/kT_2)} \quad . \quad . \quad . \quad (27)$$

or better, using the Schwarzschild temperature for the reversing layers

$$\frac{T_2}{T_1}-1=\frac{\log(g_2/g_1)-\log(T_2/T_1)}{2^{\frac{5}{4}}(\chi/kT_2)} \quad . \quad . \quad . \quad (28)$$

From what has been said, the theoretical formula (28) for the temperature difference between giants and dwarfs of the same type appears fairly reliable. It contains no arbitrary constants, for χ is determined by the lines whose maxima fix the type. In applying (28) it is sufficient to use an approximate value of T_2/T_1 for the small logarithmic term on the right-hand side.

§ 9. *Mathematical Analysis of the Equilibrium of a Photosphere.*— We proceed to prove the result assumed in § 4, and to determine the law of density-distribution, etc., near the boundary of the photosphere.

* R. H. Fowler and E. A. Milne, *M.N.*, **84**, 508, 1924. Equation (24) is capable of considerable refinement when regard is paid to the actual energy-levels of the spectra concerned.

The first part of our analysis turns out to be almost identical with some recently published by Jeans,* but our application is different.

We measure x inwards. The acceleration due to the net flux of radiation σT_1^4 being $\kappa\sigma T_1^4/c$, the equation of mechanical equilibrium is

$$\frac{dp}{dx}=\rho\left(g-\frac{\kappa\sigma T_1^4}{c}\right) \quad . \quad . \quad . \quad . \quad (29)$$

The temperature at optical depth τ is given by

$$T^4=\tfrac{1}{2}T_1^4(1+\tfrac{3}{2}\tau) \quad . \quad . \quad . \quad . \quad (30)$$

from which we deduced equation (14), § 6,

$$\frac{dp'}{dx}=\rho\frac{\kappa\sigma T_1^4}{c} \quad . \quad . \quad . \quad . \quad (31)$$

which, we repeat, is valid up to the boundary to the same extent that (30) is valid. From (29) and (31) we have

$$\frac{dp}{dp'}=\frac{cg}{\kappa\sigma T_1^4}-1 \quad . \quad . \quad . \quad . \quad (32)$$

Now introduce the physical formula for κ,

$$\kappa=\xi p/p',$$

where ξ really $\propto T^{-\frac{1}{2}}$ but may be treated as a constant in the region we are discussing. We find now

$$\frac{dp}{dp'}=\frac{cg}{\xi\sigma T_1^4}\cdot\frac{p'}{p}-1=B\frac{p'}{p}-1 \quad . \quad . \quad . \quad (33)$$

say, where

$$B=cg/\xi\sigma T_1^4.$$

To solve (33) put $p/p'=\lambda$. It then becomes

$$\frac{1}{p'}\frac{dp'}{d\lambda}=\frac{\lambda}{B-\lambda(\lambda+1)} \quad . \quad . \quad . \quad . \quad (34)$$

Let λ_1, $-\lambda_2(\lambda_2>\lambda_1>0)$ be the roots of the equation

$$\lambda(\lambda+1)=B \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Then (34) integrates in the form †

$$\log p'=-\frac{\lambda_1}{\lambda_1+\lambda_2}\log(\lambda_1-\lambda)-\frac{\lambda_2}{\lambda_1+\lambda_2}\log(\lambda_2+\lambda)+\text{const.} \quad . \quad (36)$$

At the boundary $p=0$, so that $\lambda=0$, whilst $p'=\frac{1}{3}aT_0^4=\frac{1}{6}aT_1^4=p'_0$, say. Hence

$$\log\frac{p'}{p'_0}=-\frac{\lambda_1}{\lambda_1+\lambda_2}\log\left(1-\frac{\lambda}{\lambda_1}\right)-\frac{\lambda_2}{\lambda_1+\lambda_2}\log\left(1+\frac{\lambda}{\lambda_2}\right) \quad . \quad (37)$$

* *M.N.*, **85**, 199, 1925.

† This relation was first given by J. Woltjer, *B.A.N.*, 2, 171 (No. 66), 1924, in a discussion of Eddington's approximation "$\kappa\eta$" constant in the interior of a star. It has also been given by Jeans, *loc. cit.*

As we penetrate the photosphere, p' rapidly increases, ultimately to values very large compared with p'_0. Hence λ, initially zero, must rapidly increase to a value very close to λ_1. In fact it is easily found that λ is practically indistinguishable from λ_1 as soon as p'/p'_0 exceeds 6. Thus λ or p/p' attains the limit λ_1 in the immediate interior—the result stated in § 4.

We observe that $p/p'=\lambda_1$ is itself an exact solution of (33). For the condition that this should be so is simply

$$\lambda_1(\lambda_1+1)=B,$$

which is satisfied, since λ_1 is a root of (35). Moreover, the λ_1 defined in this analysis is seen from equation (7) to be identical with the λ_1 used in § 4. The solution $p/p'=\lambda_1$ is a solution containing no arbitrary constants, and not included in the general solution (36). It is, therefore, a singular solution. We have thus proved that any solution, whatever the value of the arbitrary constant p'_0, passes asymptotically into the singular solution in the immediate interior. In effect in § 4 we have been employing the singular solution as an approximation to the actual solution.

Equation (37) fixes λ as a function of p'. Hence p is known, from $p=\lambda p'$, and T is known in terms of p' from $T^4/T_0{}^4=p'/p'_0$, with $T_0{}^4=\frac{1}{2}T_1{}^4$. Thus (37) is sufficient to determine p as a function of T. Tables I. and II. below contain simultaneous values of p and T calculated in this way for the sun and for Capella, using the various constants already given in this paper. Column 1 gives the ratio of gas-pressure to radiation-pressure, which is taken as an independent variable. Column 2 gives the values of p'/p'_0, calculated from (37), and column 3 gives the optical thickness τ, determined from $p'/p'_0=1+\frac{3}{2}\tau$. Multiplying the values of p'/p'_0 by p'_0 we obtain p', given in column 4, and multiplying by λ we have p. Column 6 gives the temperature. The pressures p and p' are given in dynes per sq. cm. To convert to atmospheres they must be multiplied by 10^{-6}. The calculations refer throughout to calcium.

It remains to find the actual heights in kilometres to which these pressures and temperatures correspond.

Table I.

The Sun. $\lambda_1=8{\cdot}80$, $\lambda_2=9{\cdot}80$. $T_1=6000°$. $p'_0=\frac{1}{6}aT_1{}^4=1{\cdot}64$ dyne cm.$^{-2}$.

$\lambda(=p/p')$.	p'/p'_0.	τ.	p'.	p.	T.
					°
0	1·000	0	1·64	0	5040
3	1·057	0·038	1·74	5·21	5110
5	1·120	0·080	1·84	9·20	5180
7	1·596	0·397	2·62	18·34	5660
8	2·275	0·850	3·73	29·9	6180
8·4	3·13	1·42	5·13	43·2	6700
8·7	5·96	3·31	9·78	85·0	7870
8·75	8·24	4·83	13·5	118	8530
8·79	17·66	11·11	29·0	255	10340

TABLE II.

Capella. $\lambda_1=1{\cdot}033$, $\lambda_2=2{\cdot}033$. $T_1=6000°$. $p'_0=\frac{1}{6}aT_1^4=1{\cdot}64$ dyne cm.$^{-2}$.

$\lambda(=p/p')$.	p'/p'_0.	τ.	p'.	p.	T.
					°
0	1·000	0	1·64	0	5040
0·5	1·104	0·069	1·81	0·91	5170
0·8	1·377	0·251	2·26	1·81	5460
0·95	1·884	0·589	3·09	2·94	5920
1·00	2·547	1·031	4·18	4·18	6370
1·025	4·09	2·06	6·72	6·88	7170
1·030	5·70	3·13	9·36	9·64	7780
1·032	8·26	4·84	13·6	14·6	8550
1·0327	12·39	7·59	20·3	21·0	9450

To do this, substitute $\rho=(1-\theta)m'p/k\mathrm{T}$ in (31).* It may then be written in the form

$$\frac{1}{p'}\frac{dp'}{dx}=\frac{\xi\sigma T_1^4}{c}\cdot\frac{(1-\theta)m'}{k\mathrm{T}}\left(\frac{p}{p'}\right)^2 \quad . \quad . \quad . \quad (38)$$

By means of (34) and the relation $p/p'=\lambda$, we can eliminate the pressures, obtaining

$$\frac{\mathrm{B}d\lambda}{\lambda[\mathrm{B}-\lambda(\lambda+1)]}=\frac{(1-\theta)m'g}{k\mathrm{T}}dx \quad . \quad . \quad . \quad (39)$$

The left-hand side can be put into partial fractions and integrated. For simplicity in a first approximation, in integrating the right-hand side we replace T by some mean value, which we shall take to be T_1. We then find

$$\log\lambda-\frac{\lambda_2}{\lambda_1+\lambda_2}\log(\lambda_1-\lambda)-\frac{\lambda_1}{\lambda_1+\lambda_2}\log(\lambda_2+\lambda)=\frac{(1-\theta)m'g}{k\mathrm{T}_1}x+\text{const.} \quad (40)$$

Equations (37) and (40) combined with $p/p'=\lambda$ suffice to determine p and hence ρ as functions of the depth x.

The best way to exhibit the kind of density-distribution implied by these equations is to give some actual numerical values. The constant in (40) depends on the level from which x is measured. We shall choose this to be the level at which $p=10^{-7}$ atmos. $=0{\cdot}1$ dyne cm.$^{-2}$. We have seen in § 7 that this level corresponds very closely to the apparent limb in the case of the sun—the level at which a tangential column 10^5 kms. in length is still transparent. Here p' is practically p'_0, and λ is therefore $0{\cdot}1/p'_0$, which for $T_1=6000°$ is 0·061. The constant in (40) is thus readily found. Tables III. and IV. give the heights in kms., measured from the level 10^{-7} atmos., corresponding to certain of the entries in Tables I. and II.

* The factor $(1-\theta)$ converts the total pressure p into the partial pressure of the atoms. In all applications in this paper $1-\theta$ is taken as $\frac{1}{2}$.

Table III.

The Sun.

	λ.	x (kms.).	p (atmos.).	T.
Limb	0·061	0	10^{-7}	5040°
Upper reversing layer	3	36	5×10^{-6}	5110
Photosphere	8·75	67	$1{\cdot}2\times10^{-4}$	8530

Table IV.

Capella.

	λ.	x (kms.).	p (atmos.).	T.
Limb	0·061	0	10^{-7}	5040°
Upper reversing layer	0·5	910	$0{\cdot}9\times10^{-6}$	5070
Photosphere	1·032	2730	$1{\cdot}4\times10^{-5}$	8550

These have been selected somewhat arbitrarily, but for convenience in referring to them I have named them "upper reversing layer" and "photosphere." The former of these must not be taken too literally. We are exploring the theoretical thinning out of a mass of gas composed of calcium atoms with a general coefficient of absorption given by the Eddington-Kramers formula, ignoring all line-absorption. I use the term "upper reversing layer" to denote a level in this mass of atoms at which the pressure is neither too big for complete opacity to have developed nor too small for a tangential column to be completely transparent. It is at this level, presumably, that line-absorption becomes dominant.

We see from Table III. that in the sun the transition layer from complete transparency to complete opacity only occupies a thickness of some 70 kms. In the corresponding giant the thickness is about 2700 kms., the ratio of the thicknesses being roughly the inverse ratio of the g's.

Observations of the sun's flash spectrum * have shown that even the low-lying gases which produce the bright lines at eclipses, and presumably therefore the fainter Fraunhofer lines in the ordinary solar spectrum, occupy a thickness of 100–400 kms. By comparison of this with the 70 kms. of Table III., we can draw a most important conclusion. The conclusion is that the gases of the reversing layer which are responsible for line-absorption must be very largely supported by radiation-pressure.

Table III. has been calculated on the basis of what *a priori* might be called the large photospheric absorption coefficient of 1140. Even this large absorption coefficient only enables $\frac{1}{10}$ of the weight of the photospheric layers to be supported by radiation-pressure, and even so supported the transition layer from complete transparency to complete opacity is condensed into the comparatively narrow thickness of 70 kms. If then layers of gas stand out above the level of complete opacity by some hundreds of kilometres, it can only be in virtue of some

* *E.g.* Mitchell, *Astrophys. Journ.*, **38**, 407, 1913.

additional sustaining force.* Such a force is probably provided by the radiation absorbed in the production of Fraunhofer lines. What a general absorption coefficient of the order of 1000 is incapable of doing, a selective absorption coefficient of the order of 10^8 may easily accomplish. In a recent paper,† "The Equilibrium of the Calcium Chromosphere," I have shown in detail how this may come about for Ca^+ atoms in virtue of the H and K lines. For Ca^+ atoms, all the atomic circumstances are known, and a fairly rigorous discussion was possible. But it seems probable that an equilibrium of something of the same character may hold for all the atoms which take part in the formation of absorption lines, though it may in some cases be of the "partially supported" rather than the "fully supported" type.‡

Further discussion of the reversing layer would be alien to the subject of this paper. But it seems desirable to insist that Table III., combined with our knowledge of the flash spectrum, is strong evidence for the importance of radiation-pressure in the reversing layer. Naturally, Table III. must not be taken to mean that the actual pressures at the levels corresponding to the first two rows are as calculated; it merely states the pressures to be expected in the absence of line-absorption. The third row is, however, on a different footing, for here opacity is established, and it seems probable that it gives a correct idea of the actual state of things at this level.

§ 10. *The Temperature Gradient.*—A glance at Tables III. and IV. shows that the temperature gradient in the transition layer from λ small to λ near λ_1 is of the order of 100° per km. for the sun, 2° per km. for Capella. Gradients of the same order of magnitude are found for the immediate interior. The gradient for Capella is of the order of that found by Eddington for the far interiors of stars. The gradient for the sun is large in comparison.

But this comparatively high gradient in the sun is inevitable unless radiation-pressure in the solar atmosphere is far larger than we have supposed, which would require values of κ larger even than those we have found. This is independent of any theory about κ. It only involves the assumption that the immediate interior is in radiative equilibrium and that p/p' is constant there; and we have seen that the former of these assumptions really follows from the observed darkening over the solar disc. To see this, we have from first principles, if p/p' is constant,

$$\frac{dp'}{dx}=\frac{p'}{p+p'}g\rho.$$

Inserting $p(1-\theta)/\rho=k\mathrm{T}/m'$, $d\mathrm{T}/\mathrm{T}=\frac{1}{4}dp'/p'$, we find

$$\frac{d\mathrm{T}}{dx}=\frac{\frac{1}{4}p}{p+p'}\cdot\frac{m'g(1-\theta)}{k},$$

* Above the transparency level κ is so small that radiation-pressure as arising from general absorption is negligible. Apart from line absorption, then, the density-distribution would be of the heavily-condensed purely gravitational type.

† *M.N.*, **85**, 111, 1924.

‡ In another paper in this number of the *M.N.*, p. 739 ("The Widths of Fraunhofer Lines"), I have shown that a similar conclusion follows from a study of the observed contours of absorption lines.

or
$$\frac{dT}{dx}=\frac{\frac{1}{4}\lambda_1}{1+\lambda_1}\cdot\frac{m'g(1-\theta)}{k} \quad . \quad . \quad . \quad . \quad (41)$$

As radiation-pressure becomes insignificant, $\lambda_1/(1+\lambda_1)\to 1$, and thus $\frac{1}{4}m'g(1-\theta)/k$ is an upper limit to the temperature gradient.

For the sun, on our theory of κ which requires the large value $\kappa=1140$ in the immediate interior, λ_1 is 8·8 and $\lambda_1/(1+\lambda_1)=0{\cdot}9$. The gradient given by (41) then comes to 150° per km.* For Capella, λ_1 is 1·033, $\lambda_1/(1+\lambda_1)$ is 0·51, and the gradient comes to 2° per km.

Formula (41) is independent of any physical assumption about κ. Smaller gradients can only be obtained by taking a smaller value of λ_1. But from (5), we have always in the immediate interior
$$\kappa=\frac{1}{1+\lambda_1}\cdot\frac{cg}{\sigma T_1^4},$$
so that smaller values of λ_1 require even bigger values of κ.

In the transitional region, before λ becomes constant, the form of (41) is found to be
$$\frac{dT}{dx}=\frac{\frac{1}{4}\lambda^2}{\lambda_1(\lambda_1+1)}\cdot\frac{m'g(1-\theta)}{k} \quad . \quad . \quad . \quad (42)$$
and so the gradient tends to zero at the boundary, as is obvious, otherwise since T tends to a limit.

§ 11. *The Fit of the "Immediate Interior" on to the "Far Interior." Comparison with Eddington's Calculations.*—We have treated ξ as a constant. Actually it varies approximately as $T^{-\frac{1}{2}}$, and thus decreases towards the interior. With it decreases λ_1, the asymptotic value of p/p'. We have seen that p/p' rapidly approaches λ_1, assuming λ_1 to be an absolute constant. It is reasonable to conjecture that if we solved equation (34), taking account of the variability of ξ (or B), the solution λ would continually strive to approach the local asymptotic value λ_1 calculated from the local value of T. So to speak, we should anticipate that the actual solution would continually "hunt" the local asymptotic solution. It is substantially this result which is proved in Jeans' recent paper,† and we shall be content to assume it.

Refinements of our analysis which would take into account this factor $T^{-\frac{1}{2}}$ and the similar factor T^{-1} in (39) will suggest themselves to the reader. For example, according to Table I., the temperature is already over 8000° at the complete-opacity level, $\tau=4{\cdot}5$, so that a mean temperature somewhat higher than 6000° should have been taken in evaluating ξ. But in this paper, in which it is mainly orders of magnitude which are insisted on, it has not seemed desirable to introduce such refinements.

We now see that the solution for the immediate interior will gradually pass into Eddington's solution for the far interior, in virtue of the slow change of ξ. The ratio λ_1 will slowly pass into Eddington's $\beta/(1-\beta)$ for the interior, by decreasing values.

* This large gradient does not endanger the stability of the equilibrium. By a well-known argument, $p\propto T^4$ is stable provided $\gamma>\frac{4}{3}$, and for a monatomic gas γ, the ratio of the specific heats, is $\frac{5}{3}$.

† *M.N.*, **85**, 196, 1925.

A word as to the relation of our mathematics to Eddington's assumption "$\kappa\eta$=const." may be of interest. Our equation (32) becomes exactly Eddington's equation for the interior when we replace $cg/\sigma T_1^4$ by $4\pi cGM/\eta L$, to which it is equal near the boundary. Eddington solves the equation

$$\frac{dp}{dp'}=\frac{4\pi cGM}{L}\cdot\frac{1}{\kappa\eta}-1 \quad . \quad . \quad . \quad . \quad . \quad (43)$$

on the assumption $\kappa\eta$=constant. It follows from this that ρ/T^3 is constant, and since physical theory gives $\kappa\propto\rho/T^{\frac{7}{2}}$, we shall have a solution corresponding to the facts if $\eta\propto T^{\frac{1}{2}}$.

But the converse of this is not true.* The solution ρ/T^3=const. is not the only solution corresponding to $\kappa\propto\rho/T^{\frac{7}{2}}$, $\eta\propto T^{\frac{1}{2}}$. In fact if we substitute $\kappa\propto\rho/T^{\frac{7}{2}}$, $\eta\propto T^{\frac{1}{2}}$ in (43) we do not recover the solution ρ/T^3= constant. The equation reduces to (33), of which the solution is (36). This solution shows that if p/p' or, what is the same thing, ρ/T^3 possesses at any point one special value,† it will always possess that value ; if it does not, it will strive to approach this value as we penetrate inwards. Thus Eddington's solution for the far interior is a singular solution, analogous to the singular solution we have used as an approximation for the immediate interior. In the far interior the actual solution must always be very close to the singular solution. But near the boundary, we can only fit the actual boundary conditions $p_0=0$, $p'_0\neq 0$ by abandoning the singular solution and using the actual solution.

§ 12. *General Discussion.*—The correctness or otherwise of the results of this paper turns largely on the extent to which the Kramers-Eddington formula for the absorption coefficient of a highly ionised gas at a temperature of several million degrees can be used for a moderately ionised gas at a temperature of a few thousand degrees. The absorption coefficients correspond in the first case to the soft X-ray region, in the second case to the optical region.

The proportionality of κ to $P_e/T^{\frac{7}{2}}$ seems inevitable on any theory, and the only question arises as to the constant of proportionality. After incorporating the odd $T^{-\frac{1}{2}}$ into this constant we have called it ξ. The following considerations are relevant :—

1. A somewhat bigger ξ seems to be required to give the correct value for the stellar absorption coefficient in the far interior.

2. Our value for ξ appears to give the right order of magnitude for the absorption coefficient beyond a series limit as a function of frequency.

3. It gives a surprisingly large opacity for the photospheric layers, suggesting that ξ may be too large, but

4. The ratio of radiation-pressure to gravity, even so, is only about twice its value in the far interior.

5. The high opacity gives a rather low value, 10^{-4} atmos., for the pressure at which opacity sets in. Higher pressures would require a smaller ξ.

6. It gives rather a high value, 10^{-7} atmos., for the pressure at the

* Substantially, this argument is due to Woltjer and Jeans, but it seems worth while restating in this context.

† The special value of p/p' is of course λ_1, the positive root of (35).

limb, *i.e.* the pressure at which a long tangential column is just transparent. Lower pressures would require a larger ξ.

7. The transition layer from complete transparency to complete opacity in the sun is about 70 kms. A greater thickness might have been expected, but this would require lower pressures, therefore higher radiation-pressure, therefore an increased ξ.

In view of these considerations, and in view of the very reasonable orders of magnitude of the predicted pressures, it is impossible to say that ξ is systematically too high or too low. The inference is that the photospheric opacities do have the high values assigned to them by the formula.

Some means of testing in the laboratory the Kramers-Eddington formula for the general opacity of an ionised gas is greatly to be desired.

§ 13. *Summary.**—1. The purpose of the investigation is to apply to the photospheric gases the formula for the general opacity of an ionised gas used by Eddington for the far interior of a star and deduced by him from Kramers' theory of X-ray absorption.

2. The determination of the absorption coefficient κ in the photospheric layers involves the determination of the ratio of radiation-pressure to gas-pressure, which has not the same value as in the far interior. This again depends on κ. The interrelationship leads to a quadratic equation for κ, the coefficients depending only on the surface gravity g and the effective temperature T_1 of the star, and the atomic constants of the material composing the photosphere (Equation (11)).

3. The ratio of gas-pressure to radiation-pressure, as arising from general absorption, is zero at the boundary but rapidly attains an approximately constant value in the immediate interior.† Radiation-pressure is about twice as important in the immediate interior as in the far interior. The values of "$1-\beta$" are 0·102 and 0·493 for the immediate interiors of the sun and Capella. Eddington's values for the far interiors are 0·050 and 0·283.

4. The mass absorption coefficient is zero at the boundary, but rapidly attains an approximately constant value in the immediate interior. The values for the immediate interiors of the sun and of Capella are about 1140 and 133, taking calcium as a typical constituent. In the case of the sun such an opacity is over twenty times Eddington's typical absorption coefficient for the interior. It is several thousand times the value calculated by Russell and Stewart from thermal absorption and scattering by free electrons in the photosphere. If the application of the Eddington-Kramers formula is justifiable, it is evident that photo-electric absorption is the dominant cause of general absorption in a stellar atmosphere.

5. With these coefficients, complete opacity is established at $1{\cdot}1\times10^{-4}$ atmos. for the sun, at $1{\cdot}3\times10^{-5}$ atmos. for Capella. A simple formula, (18), is found for the pressure in any star at the opacity-level. This level can be considered to mark the base of the reversing layer.

6. For the sun, a column of gas 10^5 kms. long becomes completely transparent at 10^{-7} atmos. This pressure should therefore correspond

* The content of § 2 is not included in this summary, as it stands by itself.

† This result has also been given by Jeans.

to the apparent limb of the sun, apart from complications due to the consequences of line-absorption in the reversing layer.

7. For the sun, the transition from 10^{-7} atmos. to 10^{-4} atmos. occupies about 70 kms. For Capella, from 10^{-7} atmos. to 10^{-5} atmos. occupies 2700 kms.

8. Since the gases producing line-absorption in the reversing layer appear to be in general thicker than 70 kms., the conclusion is that they are largely supported by the additional radiation-pressure arising from line-absorption. If we have overestimated the general opacity of the photospheric layers, the photospheric layers would be more condensed still, and the conclusion becomes stronger.

9. A revised calculation of the temperature-difference between a giant and dwarf of the same spectral type gives $T_2-T_1=600°$ for type Go. The pressures in the atmospheres of a giant and a dwarf of the same type should be approximately in the ratio of the square roots of the values of gravity. An expression for the temperature-difference is given, formula (28).

10. Tables are given for the temperatures, pressures, and ratios of gas-pressure to radiation-pressure at various levels in the sun and in Capella.

11. The temperature-gradient in the immediate interior of the sun is about 150° per km., of Capella 2° per km.

12. Finally, the theory appears to lead to a reasonable picture of the photospheric layers of a star such as the sun, and the predicted orders of magnitude of the pressures are in general agreement with those deduced from observations of spectra. The theory involves no empirical constants. The inference is that the Eddington-Kramers formula for the general opacity of a gas does in fact hold, as regards order of magnitude, for the gases composing the photospheric layers of a star, and that almost the whole of the opacity arises from photo-electric emissions of electrons from atoms and from absorptions by free electrons when in collision with atoms.

On the Masses, Luminosities, and Surface-Temperatures of the Stars. (*Final Note.*)

By J. H. Jeans.

1. On account of the astronomical importance of the subject, it seems desirable that some reply should be made to Professor Eddington's note in the March number of the *Monthly Notices*, the more so as the question of stability which he has raised both there and in a recent letter to *Nature* (1925 March 21) merits fuller consideration than has yet been given it.

In *Nature* of 1924 May 31, Professor Eddington stated his main conclusion (the mass-luminosity law) in the form: "The result of the theoretical discussion is that we obtain for the absolute magnitude of the star a formula of the form

$$m=f(\mathrm{M})+a\ constant \quad . \quad . \quad . \quad (1)$$

where $f(M)$ is a definite function of the mass which can be tabulated, and all the doubtful constants and theoretical uncertainties are segregated in the value of the additive constant."

The above equation I believe to be entirely erroneous, my analysis showing that the true relation is

$$m = F(M) + k \log T_e + a \textit{ constant} \quad . \quad . \quad (2)$$

where T_e is the star's effective temperature and k is a constant which depends primarily on the power to which the temperature is raised in the formula for the coefficient of opacity, and to a lesser degree on other constants. In any case, k is fairly near to unity, and I have found that the best fit with observation is obtained by taking $k=1$ precisely.

In so far as their general physical interpretation is concerned, the two formulæ (1) and (2) are obviously in flat contradiction. Under (1) a star of given mass M must necessarily radiate energy away at a predetermined rate, whether it is generating sufficient energy to do so or not, while under (2) all rates of radiation are open to a star of mass M ; it can always adjust its rate of radiation to balance exactly the rate at which energy is being generated in its interior, and does this by selecting a surface-temperature such that equation (2) is satisfied. For instance, my diagram (*M.N.*, 1925 March, p. 401) shows that a star of mass half of that of the sun would attain a Bo temperature if its rate of generation of energy were that corresponding to $m=7{\cdot}6$, say 0·4 ergs a second for each gramme of its mass. Professor Eddington's view (*M.N.*, 1925 March, p. 405) is that a star of half the sun's mass which generated energy at 0·4 ergs per second per gramme would " fail to find equilibrium, and go on expanding indefinitely."

To take a concrete instance, Professor Eddington's result would seem to commit him to the view that when a star breaks up by fission each constituent must " fail to find equilibrium and go on expanding (or contracting) indefinitely," unless there is some regulation of the rate of generation of energy which arranges that this shall be liberated at precisely the amount prescribed by formula (1). My own view is, of course, that each constituent attains equilibrium by adjusting its surface-temperature to the appropriate value. In general, this adjustment requires a great increase of temperature ; and this may, I think, account for the high observed temperatures of newly formed binaries, a view which is further developed in the paper immediately following this. A star which has undergone fission an abnormally large number of times ought to have an abnormally high temperature for its mass, a consideration which may possibly be found to provide an explanation of the existence of the " white dwarfs."

2. In place of pointing out in his " reply " any error in the analysis by which my equation (2) is obtained, Professor Eddington merely claims that he could have reached my result in a shorter way. His way is certainly shorter, but it disregards entirely the difficult question of the boundary condition at the star's surface, and it fails either to show that a solution exists or that the solution obtained is unique.

What, in effect, it really does is to check the accuracy of my value for the coefficient k.* In confirming my value for k, Professor Eddington destroys his equation (1). In place of bringing him to his mass-luminosity relation, Professor Eddington's analysis now brings him to "the common-sense view that a star of mass M, generating energy at a given rate E, can always adjust its radius and surface-temperature so as to be in equilibrium."

3. Professor Eddington, however, continues: "Unfortunately mathematical analysis shows plainly that the common-sense view will not work. For if the given rate of generation of energy is greater than the rate of radiation L fixed by the formula, the energy of the star is increasing, and it must expand. By (3) L varies as R^{-n}, and therefore diminishes. This augments the discrepancy. Accordingly, the star fails to find equilibrium and goes on expanding indefinitely."

There appear to me to be two errors in this argument, one of form and one of substance. As regards the error in form, the second sentence assumes the star not to be in equilibrium (the star's energy is increasing, and it is expanding), while the sentence following assumes the star to be in equilibrium, for formula (3) which is quoted refers only to stars in equilibrium, and in any case we have no knowledge as to the dependence of the rate of radiation of a star on its radius except when there is equilibrium. Further, Professor Eddington's argument assumes without proof that, in the circumstances stated, the star would expand; in actual fact I believe, and hope shortly to prove (§ 4), it would contract. The error first mentioned prevents the argument from proving anything; but if it did prove anything, it would (on replacing expansion by contraction, etc.) prove precisely the opposite of what it set out to prove. It may be noticed further that the argument, if it had been valid, would not only have destroyed my configurations (2) (at which I imagine it was aimed), but also Professor Eddington's configurations (1), and indeed all the stars in the sky. It is a double-edged weapon which would have the most devastating effects, except for the two fatal errors just mentioned.

4. The proper treatment of the question of stability is, I think, on the following lines:

So long as we limit ourselves to spherical configurations, equation (2) gives all possible configurations of equilibrium. Every term in equation (2) is a single valued function of a single variable; so that when these configurations are arranged in linear series, in the manner of Poincaré, there can be no intersections and, consequently, no points of bifurcation. It follows that there can be no change of stability on a linear series. If a star is spherical and stable at its creation, it will re-

* That is to say, Professor Eddington checks the value in terms of unknown constants, although preferring to give to these constants values which make $k = 2$ in place of my $k = 1$. The value $k = 2$ is probably within the limits permitted by observation. He had previously arrived at this value $k = 2$ by a separate investigation, but even with $k = 2$ he describes the term $k \log T_e$ as "a small correction" (*loc. cit.*, p. 403). In his earlier papers k was taken to be zero, but with k equal to 1 or 2, and T_e capable of variation from 0 to ∞, $k \log T_e$ can hardly be legitimately described as a small correction.

main stable throughout its life so far as spherical displacements are concerned.

Let us now consider Professor Eddington's hypothetical case of a star originally in equilibrium and radiating at a rate L, which suddenly changes its rate of generation of energy to L+dL. For the first moment, radiation is being created at a rate L+dL, and radiated away at a rate L, so that there is an accumulation of radiation in the star which immediately upsets its equilibrium. A step further along the linear series there is a configuration of equilibrium corresponding to a rate of radiation L+dL, and the configuration of the star may be regarded as being arrived at by a small displacement from this position. In virtue of the argument given above this position is one of stable equilibrium. Hence it will be seen that a motion which satisfies all the conditions of the problem is one in which the star oscillates about this position of equilibrium, the extreme range of each constituent oscillation being that which brings the star to the configuration in which it started, namely, the configuration corresponding to radiation L.

The first move of every constituent oscillation of the star (since these all start from rest) will be toward the L+dL configuration of equilibrium, and as the radius of this configuration is less than that of the L configuration, the star's first move will consist of a contraction (not an expansion as is required by Professor Eddington's argument quoted above). If the increase in L is slow and gradual, the oscillations will, of course, be of infinitesimal amplitude, and the star will steadily contract, travelling slowly along the linear series of equilibrium configurations.

It may at first sight seem a bit paradoxical that an increase in a star's rate of radiation should cause it to contract. On tracing the matter out, this will be seen to be a consequence of the largeness of the index r in the formula for the coefficient of opacity, namely,

$$k \propto \rho / T^r \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To see the connection, let us consider a hypothetical star in which this index r is enormously large. Imagine such a star, while radiating in equilibrium at a rate L, suddenly to increase its rate of generation of radiation to L+dL. Before ρ begins to change, the effect of this excess generation of energy is to increase T. Since r in formula (3) is supposed enormously large, this depresses k almost to zero, so that the star becomes transparent, all its interior energy is radiated away and it sinks to zero temperature, thereby contracting. In actual fact, the increase in ρ and that in T would, of course, proceed simultaneously and at such relative rates as to keep k finite in formula (3).

The actual value of r is about $\frac{7}{2}$, and we can show that with this value for r the same phenomena would occur, although naturally in a less extreme form. On the other hand, had the value of r been $\frac{5}{2}$, an increase in the rate of generation would have resulted in expansion. In this case Professor Eddington's argument would still fail, since L would now increase with R.

In nature the normal process is, of course, the reverse of that we have been considering. The radiation from a star steadily decreases,

and the star experiences an expansion which may be partially or wholly counterbalanced, or more than counterbalanced, by a loss of the star's mass.

5. So far our discussion of stability has dealt only with spherical displacements. There seems to be no reason for suspecting that spheres of gas can be unstable for other than spherical displacements, but Eddington has suggested that when radiation-pressure is so powerful as almost to balance gravitation, they may, although still stable, be so near to instability that a very small amount of rotation or other disturbing cause may suffice to upset the balance (" a gas-sphere under the influence of two opposed and nearly balanced forces would probably be on the verge of instability "). He has suggested that such bodies may continually divide and subdivide until their masses are below the danger-limit, after which there would be no further cause for subdivision. In this way he attempts to explain the circumstance that the majority of stars are of masses such that gas-pressure and the pressure of radiation are comparable in their interiors.

The suggestion that the influence of two opposed and nearly balanced forces brings a body to the verge of instability cannot, I fear, survive examination. If it were true, a star of very small mass, which is a gas-sphere under the influence of the two opposed and nearly balanced forces of gravitation and gas-pressure, would be still nearer the verge of instability. Such a star ought then to break up, and the more it broke up the more unstable each constituent would be (for the smaller the mass of a star, the nearer the opposing forces of gravitation and gas-pressure approach to absolute equality), until we should be left with a universe of explosive fragments.

That the influence of pressure of radiation does not make a body more prone to rotational break-up can be seen from a study of the equations of equilibrium, or more briefly from a consideration of the end of the process of rotational break-up. Imagine a body of mass $m+m'$ ending by breaking into two detached masses which describe about one another a circular orbit of radius a. The angular velocity ω in this orbit is given by $\gamma(m+m')=a^3\omega^2$, whence the orbital angular momentum is

$$\frac{mm'}{m+m'}a^2\omega=\frac{mm'}{(m+m')^{\frac{1}{2}}}(\gamma a)^{\frac{1}{2}} \quad . \quad . \quad . \quad (4)$$

The bodies m, m' cannot remain apart with an angular momentum less than this, so that if the initial angular momentum of the system before the supposed break-up occurred had been less than this, the break-up could never have taken place. The quantity on the right hand of (4) is, however, entirely independent of the importance, and even of the existence, of the pressure of radiation, so that unless a rotating mass has enough angular momentum to break up without pressure of radiation, it will not break up at all.

Other supposed mechanisms of break-up can be treated in much the same way, the essential point being that even if pressure of radiation had some influence in the early stages of break-up, it is practically powerless to help keep the constituents separate after they have broken

up. To take a concrete instance, the gravitational attraction between the two constituents of β Lyræ is about 4×10^{37} dynes; the utmost possible value for the repulsion between them produced by the pressure of radiation is 4×10^{24} dynes. Thus pressure of radiation can at most balance one part in 10^{13} of the gravitational attraction; the remaining $10^{13}-1$ parts are balanced by the presence of angular momentum which must have been in the system from the outset, and would have broken the star up even had the pressure of radiation been non-existent.

6. The conclusion of the whole matter appears to be that pressure of radiation introduces no new factors, and is of comparatively little importance, in the problems we have had under discussion. The old-fashioned sphere of gas, in which radiation was left entirely out of account, still provides a remarkably good model of a star. It reproduces all phenomena with fidelity qualitatively, but requires modification quantitatively in two respects. In the first place, the capacity of the old model for transfer of energy from centre to surface was far too small, the transfer by radiation being millions of times as great as that by gaseous conduction; and in the second place, the capacity of the old model for heat-energy was too small, a fully ionised gas having about twenty times the heat-capacity of unionised gas of molecular weight 40. To this increase of heat-capacity a further contribution is added by the presence of radiation, ranging from about 10 per cent. in our sun to perhaps 50 per cent. in a B-type star, and about 120 per cent. in a star of mass 20. To these may be added the somewhat different circumstance, so convincingly demonstrated by Eddington, that the ionised star can attain to far higher densities than the mass of plain gas. But apart from the need for these quantitative corrections, a sphere of ordinary gas without radiation appears to provide a good model, and the various paradoxical theorems recently enunciated do not seem to me to correspond to anything in the facts of nature.

On Cepheid and Long-period Variation and the Formation of Binary Stars by Fission. By J. H. Jeans.

1. It is generally accepted that the great majority of spectroscopic and short-period binaries have been formed by fission. Newly formed binaries are distinguished by low eccentricity of orbit and by orbital dimensions so small that the two constituents are almost in contact. As was first clearly shown by Campbell, the majority of these newly formed binaries exhibit spectra of the " early " types, O, B, and A.

So long as an " early " spectral type was supposed to indicate an early stage of stellar evolution, this last circumstance admitted of a simple, although perhaps not entirely convincing, explanation; it could be taken to mean that a star broke up either very early in its career or not at all. With the abandonment of the old view of stellar evolution, this interpretation of the spectra of binaries had also to be

abandoned, and there is need for a new interpretation which shall be in conformity with new views of stellar evolution. The first part of the present paper suggests such an interpretation; the second part deals with the actual process of fission.

2. In a recent paper * I gave a diagram showing how all the theoretically possible configurations of equilibrium for a star could be represented in a plane in which a star's mass was taken for abscissa and its luminosity for ordinate. The star's effective temperature was shown to be uniquely determined by its mass and luminosity, so that all

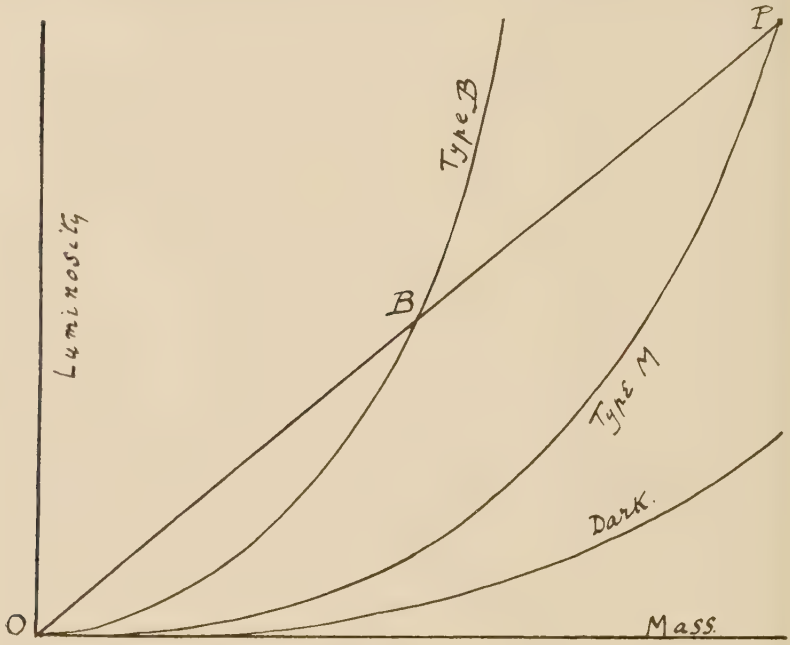


FIG. 1.

stars having the same specified effective temperature lay on a definite curve in this diagram. The curve T=0 coincided with the axis of mass, the curve T=∞ with the axis of luminosity, and the various curves representing intermediate effective temperatures lay as shown in fig. 1. Since spectral type depends almost solely on temperature, these curves may also be regarded as the loci of configurations of given spectral type.

Imagine a star of mass M, generating energy at a rate E, and so of luminosity proportional to E, to divide into two precisely similar constituents, each of mass $\frac{1}{2}$M and each generating energy at a rate $\frac{1}{2}$E. Let P in fig. 1 represent the configuration of the parent star, then the straight line OP will represent all stars for which E/M has the

* *M.N.*, 1925 Jan., p. 210.

same value as it has for this star. Thus each of the two constituents ($\frac{1}{2}$M, $\frac{1}{2}$E) will lie on this line, and will of course be represented by B, the middle point of OP. A glance at the arrangement of curves in fig. 1 will show that the effective temperature of B is necessarily higher than that of P. Or, to put the matter in terms of spectral type, the process of fission advances a star to an "earlier" spectral type. It is on these lines that it is suggested that the early spectra of newly formed binaries are to be explained, and we turn to quantitative calculations.

3. In another recent paper,* I have obtained the equation

$$m+\log T_e+11{\cdot}75\log M+32{\cdot}5\log \Psi_0=8{\cdot}332 \quad . \quad . \quad (1)$$

This determines T_e the effective temperature of a star, and so also its spectral type, in terms of its absolute magnitude m and its mass M, for Ψ_0 is a function of M alone.† To understand the meaning of equation (1) more clearly, let us replace m by its value $-2{\cdot}5\log E+$ *a constant*, where E is the rate of emission of radiation from the star. Further, since Ψ_0 is a function of M which only varies slowly with M let us put

$$32{\cdot}5\log \Psi_0=11{\cdot}75\log \theta \quad . \quad . \quad . \quad (2)$$

so that θ is a function of M which only varies slowly with M. In actual fact θ is a positive quantity which is equal to unity for stars of very small mass, is equal to 0·93 for stars equal to the sun in mass, and is equal to 0·57 for a star of five times the sun's mass. After this it decreases only very slowly, being equal to 0·36 for a star of twenty times the sun's mass, and ultimately vanishes for a star of infinite mass.

Equation (1) may now be written in the form

$$\log T_e=2{\cdot}5\log E-11{\cdot}75\log(\theta M)+a\ constant \quad . \quad (3)$$

showing that T_e is determined by the balance between the rate E at which radiation is generated in the star's interior and its mass M, or, if we prefer, its "corrected mass" θM. An increase in E increases T_e, while a decrease in M, E remaining constant, also increases T_e. If E decreases more rapidly than $(\theta M)^{4{\cdot}7}$, a star's effective temperature is decreasing, so that it is moving to a "later" spectral type, while if E is decreasing less rapidly than $(\theta M)^{4{\cdot}7}$, T_e is increasing and the star is moving to an earlier spectral type, and of course conversely as regards increases in E. On the view of stellar evolution I have recently put forward, the value of E is uninfluenced (to a first approximation at least) by the temperature and density of the star's interior, depending solely on the spontaneous decay of the stellar matter into radiation. The mass also changes as required by the spontaneous decay, so that equation (3) enables us to trace the value of the effective temperature, and consequently also the spectral type, of a star throughout the whole course of its evolution.

4. Let us now return to the simple idealised case already con-

* *M.N.*, 1925 March, p. 394.

† This is tabulated in *M.N.*, 1925 March, p. 395.

sidered, in which a star of mass M, generating energy at a rate E, divides into two precisely similar constituents each having mass $\frac{1}{2}$M and generating energy at a rate $\frac{1}{2}$E. Equation (3) gives the value of T_e for the parent star, while the value for each of the constituents after fission is T_e', given by

$$\log T_e' = 2{\cdot}5 \log (\tfrac{1}{2}E) - 11{\cdot}75 \log (\tfrac{1}{2}\theta' M) + a \textit{ constant} \quad . \quad (4)$$

where θ' is the value of θ corresponding to mass $\frac{1}{2}$M, and the constant on the right is the same as in equation (3).

On subtracting corresponding sides of equations (3) and (4), we obtain

$$\begin{aligned} \log (T_e'/T_e) &= (11{\cdot}75 - 2{\cdot}5) \log 2 - 11{\cdot}75 \log (\theta'/\theta) \\ &= 2{\cdot}785 - 32{\cdot}5 [\log \Psi_0' - \log \Psi_0] \quad . \quad . \quad (5) \end{aligned}$$

A numerical investigation shows that the right-hand member of this equation is positive for all masses, so that T_e' is always greater that T_e, confirming our inference from the diagram shown in fig. 1. Thus when a star divides into two equal and similar parts, the effective temperature of each constituent is higher than that of the parent star, so that the star appears to move towards an "earlier" spectral type.

It will, of course, be understood that this change of type is something quite distinct from the slow evolutionary change imagined by current theories of stellar evolution. The rate at which it occurs, when once the fissional process has set in, is determined solely by the rate at which fission proceeds; it has nothing whatever to do with the star's evolution in the ordinary sense. Speaking with reference to the astronomical scale of time, we can describe the change of type as being almost instantaneous.

5. The Lick "Third Catalogue of Spectroscopic Binary Stars"* contains thirty stars, apparently of recent formation, for which either $m+m'$ or $(m+m')\sin^3 i$, or both, can be determined with fair certainty. These are shown in the following table :—

Boss.	Star.	Type.	e.	$(m+m')\sin^3 i$.	$m+m'$.	m'/m.
..	TV Cas.	B9	0·0	..	2·84	0·55
145	π Cas.	A5	0·010	2·69	..	0·99
175	ν And.	B3	0·0	2·60	..	0·73
497	i Tri f	A5	0·010	1·761	..	1·00
844	o Per.	B1	0·00	9·21	..	0·70
1076	d Tau.	A3	0·0	4·0	..	0·47
1213	66 Eri.	B9	0·09	5·0	..	0·79
1301	η Ori.	B1	0·016	21·8	..	0·95
1314	ψ Ori.	B2	0·0651	9·72	..	0·76
1478	β Aur.	Aop	0·00	4·38	..	0·98
..	B.D. 6° 1309	Bop	0·035	138·9	..	0·84
2119	V Pup.	B1p	0·08	33	..	..
3476	α Vir.	B2	0·10	15·4	..	0·61

* *L.O.B.*, No. 385 (1924).

Boss.	Star.	Type.	*e*.	$(m+m')\sin^3 i$.	$m+m'$.	m'/m.
..	U Cor. Bor.	B8	0	5·7	5·99	0·38
4277	μ' Sco.	B3p	0·05	16·5	..	..
4328	ϵ Her.	A0	0·023	2·7	..	0·62
..	U Oph.	B8	0·00	9·890	..	0·88
4388	*u* Her.	B3	0·053	9·4	..	0·38
..	TX Her.	A5	0·0	3·78	3·81	0·86
4643	108 Her.	A2	0·00	2·90	..	0·69
4776	β Lyr.	B8p	small	23·41	30·57	0·45
4788	50 Dra.	A0	0·012	1·85	..	0·95
..	RS Vul.	B8	0·053	5·70	7·09	0·31
..	Z Vul.	B3	0·0	..	7·60	0·45
4947	2 Sge.	A0	0·05	1·56	..	0·71
5173	18 Vul.	A2	0·012	4·33	..	0·91
..	Y Cyg.	B2	0·0	31·57	31·9	0·92
5375	57 Cyg.	B3	0·137	3·46	..	0·93
5764	2 Lac.	B5	0·015	1·58	..	0·82
6142	..	B0	0·105	31·2	..	0·69

The average value of $m+m'$ for the seven stars for which this can be determined is 12·8⊙. For the remaining twenty-three stars $(m+m')\sin^3 i$ has the average value 13·2⊙. If we follow Schlesinger in supposing that the chance of detection of a binary is proportional to $\sin i$, then the average value of $\sin^3 i$ for all detected binaries must be 0·68, whence, neglecting any small correlation there may be between $m+m'$ and $\sin i$ (for discovered binaries), we find that the average value of $(m+m')$ for these twenty-three stars is 19·4⊙. The average value of $m+m'$ for the whole list of thirty stars is now 17·9⊙.

The values of T_e'/T_e calculated from equation (5) for different values of M (or $m+m'$) are as follows :—

$$\begin{aligned}
&M=50 \quad \log (T_e'/T_e)=0\cdot575 \quad \text{or } T_e'= 3\cdot76T_e\\
&M=25 \quad \log (T_e'/T_e)=0\cdot673 \quad \text{or } T_e'= 4\cdot71T_e\\
&M=17\cdot9 \quad \log (T_e'/T_e)=0\cdot816 \quad \text{or } T_e'= 6\cdot55T_e\\
&M=10\cdot0 \quad \log (T_e'/T_e)=1\cdot017 \quad \text{or } T_e'=10\cdot40T_e\\
&M= 5\cdot0 \quad \log (T_e'/T_e)=1\cdot417 \quad \text{or } T_e'=26\cdot12T_e.
\end{aligned}$$

6. The ideal case of a star dividing into two equal and similar masses cannot ever be realised in nature, for the circumstance that instability first sets in through the unsymmetrical third zonal harmonic introduces a dissimilarity from the outset. The average mass-ratio of the twenty-eight stars in the above table, for which m'/m is known, is found to be 0·7326, representing the fission of a mass M into constituents of masses 0·577M and 0·423M. Aitken* gives the average mass-ratio of binaries of types B to B8 as 0·70, representing fission into constituents of masses 0·588M and 0·412M. The luminosity also is divided unevenly. The larger constituent is almost invariably

* *The Binary Stars*, p. 207.

the most luminous (β Lyræ possibly providing a solitary exception), but the ratio of E/M can vary widely for the two components.

Consider now the general case of a parent star dividing so that its mass M forms constituents μM, $(1-\mu)$M between which its rate of generation of energy E is divided in amounts ηE, $(1-\eta)$E. In place of equation (5) we readily find the equation

$$\log (T_e'/T_e) = 2{\cdot}5 \log \eta - 11{\cdot}75 \log \mu - 32{\cdot}5 [\log \Psi_0' - \log \Psi_0] \quad . \quad (6)$$

The following table gives values of T_e'/T_e for different values of η and μ, the combined mass of the star being assumed to be $20{\cdot}0\odot$. The values $\mu = 0{\cdot}42$ and $0{\cdot}58$ represent average values for the two components of a binary, the values $\mu = 0{\cdot}33$ and $0{\cdot}67$ represent quite extreme values, and the value $\mu = 0{\cdot}50$ of course represents the ideal case already considered of division into two equal parts.

$\mu =$ / $\eta =$	0·33	0·42	0·50	0·58	0·67
0·33	22·9	5·89	*2·34*	*1·00*	*0·48*
0·42	40·7	10·5	*4·17*	*1·78*	*0·85*
0·50	*63·1*	*16·2*	6·31	*2·75*	*1·32*
0·58	*91·2*	*23·4*	*9·33*	3·98	1·91
0·67	*128·8*	*33·1*	*13·2*	5·62	2·69

Some of the entries obviously correspond to cases which never occur in nature: in particular, all those printed in italics violate the condition that the more massive constituent is to be the brighter.

Shapley * has found that for the average binary the ratio between μ and η is expressed approximately by the equation

$$2\mu = 0{\cdot}4 + 1{\cdot}2\eta \quad . \quad . \quad . \quad . \quad (7)$$

The values of T_e'/T_e which occur when μ and η are connected by this relation are found to be as follows:—

	$\mu =$	0·33	0·42	0·50	0·58	0·67
	$\eta =$	0·21	0·37	0·50	0·63	0·77
$(M = 20{\cdot}0\odot)$	$T_e'/T_e =$	11·5	7·59	6·31	4·90	3·89
$(M = 10{\cdot}0\odot)$	$T_e'/T_e =$	27·5	16·6	10·5	7·08	5·25

We see that the less massive constituent has the "earlier" spectral type. This is a consequence solely of Shapley's suggested law (7) and of our equation (1): it is not in any sense dependent on any special hypothesis as to the origin of binary stars.

7. It may be thought that all our calculated values for T_e'/T_e are impossibly high. It has to be admitted that they are very sensitive to changes in the value of the coefficient of log T in equation (1). We have taken this coefficient to be unity, but a value 2 is not altogether inadmissible. The effect of changing the coefficient from 1 to 2 would be to halve all our calculated values for $\log T_e'/T_e$, and so to replace all values of T_e'/T_e by their square roots. The values of T_e'/T_e is also

* "A Study of the Orbits of Eclipsing Binaries," *Princeton Contribution*, No. 3 (1915).

calculated subject to the supposition that the rate of generation of energy in the two constituents after fission is equal to that in the parent star before fission. Our values for T_e'/T_e would require correction if the process of fission affected the rate of generation of energy. We ought, perhaps, to interpret our figures merely as showing that there are ample possibilities for fission to raise the temperature of a star up to that of types B and A.

8. It is of course possible to work the tables backwards and compute the state before fission of the binary stars shown in the table of § 5. A reconstruction of the states of known binaries before fission took place shows that their states during fission must have occupied something like the range occupied by the long-period and Cepheid variables. This inevitably suggests that the long-period and Cepheid variables may be stars in process of fission into binaries. The hypothesis is by no means free from difficulties, but it would seem to be well worthy of discussion, and the remainder of the present paper is devoted to a preliminary discussion of the hypothesis.

9. The hypothesis as to the physical nature of Cepheid variables which at present holds the field is the well-known "pulsation theory," which was first brought into favour by the arguments of Plummer,* Shapley,† and others, and to which precision has been given by the mathematical investigations of Eddington.‡ In so far as any hypothesis of the nature of long-period variables holds a corresponding position, it is perhaps that of Merrill, which supposes the light variation to be the consequence of a periodic condensation of clouds in the star's atmosphere, and their subsequent dispersal by the radiation imprisoned by them.

For the purpose of mathematical treatment, Eddington treats a Cepheid as being spherical and devoid of rotation, the period of light-variation being supposed to represent the period of a natural free pulsation of the star. Merrill's hypothetical condensations and dispersals must be accompanied by a rhythmical heating and cooling of the outer layers at least of the star, and so by a pulsation of the star. It is perhaps more natural to regard this pulsation as the fundamental phenomenon, so that again the period must be that of a natural free pulsation.

Any such interpretation of the periodicity appears to be open to fatal objections. Taking the widest possible view of its structure and mechanism, a spherical star at rest will possess a certain number of possible configurations of equilibrium, and these can be divided perfectly sharply and definitely into stable and unstable configurations. The characteristic of a vibration about an unstable configuration is that its amplitude increases without limit until finally the star settles down to some entirely different, and stable, configuration of equilibrium. The characteristic of a vibration about a stable configuration is that its amplitude is gradually damped by viscosity and other dissipative forces until it falls to zero. The dynamics of oscillations of a non-rotating system has no knowledge of finite oscillations maintained at a constant amplitude.

* *M.N.*, Vols. **73–75.**
† *Astrophys. Journ.*, **40** (1914), 448.
‡ *M.N.*, **79** (1918), 1.

It is somewhat difficult to estimate what would be the rate of decay of the pulsations of a gaseous star. If γ is an average coefficient of viscosity for its interior, the time in which viscosity would reduce the amplitude of a vibration by one-half is of the order of $R^2\rho/\gamma$ or again of $M/4\gamma\rho$. The radii and masses of a number of Cepheids have been tabulated by Eddington.* If we take $M=5\odot$ and $R=2\times10^{12}$ (or $M=10\odot$ and $R=4\times10^{12}$), the value $\gamma=0{\cdot}0002$ would give a time of about 2×10^9 years for the reduction of an oscillation to half its amplitude. But the value $\gamma=0{\cdot}0002$, which is appropriate to an ordinary gas under atmospheric conditions, is far too low for the ionised gas which we believe to constitute a star's interior. The transport of momentum in such a gas results mainly from the motion of the atomic nuclei, and at 10^7 degrees C. these have a velocity of about two hundred times the velocities at ordinary atmospheric temperature. Thus our value ought to be increased some two hundredfold, thereby reducing the time in question to about 10^7 years. Judging from the number of Cepheids in the sky, this probably gives too short a life to the pulsations of a Cepheid. A more serious objection, however, is that a mass of gas has a large number of free vibrations whose periods are, in general, incommensurable. The free pulsations of a sphere of gas could only exhibit regularly recurring maxima in the very improbable event of all the vibrations except one having been damped to zero amplitude, while this particular one persisted with vigour. Finally, if the Cepheids were masses of gas whose pulsations were dying down, we ought to find pulsations of all amplitudes from zero upwards. In actual fact the light-variations cluster, with comparatively little dispersion, around a mean of about 0·7 m.; on the pulsation theory this would only be intelligible if all the pulsations had been set going at about the same time and with about equal amplitudes.

10. When we pass to systems in motion, and in particular to rotating systems, we no longer find that configurations of equilibrium or of steady motion can be sharply divided into stable and unstable configurations. For a rotating system there is an intermediate class of configurations which are " ordinarily " stable, but are secularly unstable, and a rotating system can execute oscillations of permanently finite amplitude about such a configuration. As the suggestion of the present paper is that these are precisely the oscillations observed in variable stars, it may be worth giving a short account of the main conclusions of the theory of secular instability.

A system oscillating about a state of rest keeps its amplitude of oscillation constant except in so far as this amplitude is altered by dissipative forces. In this case the effect of dissipative forces is invariably to lessen the amplitude of vibration. The reason is simple: dissipative forces use up energy and the only source from which energy can be fed to them is the energy of the oscillation itself.

The case is different with a system oscillating about a state of dynamical steady motion or about a position of relative equilibrium while in rotation. Here dissipative forces may either decrease or increase the amplitude of oscillation. A new source of energy, from

* *M.N.*, **79** (1918), 4.

which the dissipative forces can be fed, is present in the motions of the constituent bodies of the system, or in the case of a single rotating body, in the rotation itself. There is no automatic regulation of the rate at which energy is fed to the dissipative forces; it may be either faster or slower than the rate at which they consume energy, so that the amplitude of a vibration may either increase or decrease.

11. A good instance of secular instability was provided, although unconsciously, by Sir G. Darwin in his investigations on tidal friction. Let us consider two bodies such as the earth and moon rotating about one another in orbits which are initially circular. The position of the moon is fixed by two co-ordinates r, θ, and in the case we are considering r and $\theta-\omega t$ (which is the value of θ referred to rotating axes) have constant values, say a and e. If there are no dissipative forces, the moon, if knocked slightly out of its orbit, will describe a new somewhat elliptic orbit and will repeat this orbit indefinitely. Its motion can now be regarded as one of oscillation about a new circular orbit (a', ω'), so that if a' and ω' are suitably chosen the two co-ordinates r and $\theta-\omega' t$ will both be performing small oscillations about values a', e'. In the absence of dissipative forces these oscillations remain permanently of the same period; we may notice that they both have the same frequency as the orbit about which they oscillate or, if we prefer, the same frequency as the rotation of the axes to which they are referred.

If, however, the dissipative forces of tidal friction are in play, the oscillations do not gradually decrease in amplitude, as might perhaps *à priori* be expected. As Darwin showed, they both increase, the moon describing an orbit of ever-increasing radius and of ever-increasing eccentricity. The dissipative forces are in fact being fed from the earth's energy of rotation, and as soon as this source is used up, the increase in the oscillations ceases.

12. An instance of a simpler kind has been worked out by Lamb.* Imagine a small spherical particle placed inside a spherical bowl of radius a which is rotating with angular velocity ω about a vertical axis. If there is no friction between the bowl and the particle, there is a stable position of equilibrium at the lowest point of the bowl, and if the particle is displaced from this position, it will execute oscillations of constant amplitude about it. But if the interior of the bowl is rough, the particle refuses to stay in this equilibrium position; it picks up the motion of the bowl and describes spirals of ever-increasing radius about the lowest point of the bowl until it ends by describing permanently a circle at height $a-g/\omega^2$ above this point, at the same speed as the bowl. This is the same circle as would be described by a conical pendulum rotating with angular velocity ω. Lamb shows that in its earlier stages the motion of the particle may be resolved into two circular motions in opposite directions, that whose motion is opposite to the bowl being gradually extinguished, while that whose motion is with the bowl gradually increases in radius until it attains to the value already stated.

13. Returning to astronomical problems, it is known that a mass of rotating matter, which has not got too great a central condensation of

* *Proc. R.S.*, **80** (1907), 170.

mass, becomes secularly unstable as soon as it passes a certain clearly defined point on the Jacobian series of ellipsoids. The oscillation through which instability first sets in is one in which the surface of the ellipsoid undergoes a third harmonic deformation. The star begins by being pear-shaped, the extent of divergence from the ellipsoidal shape being oscillatory in amount, but showing a general tendency to increase. Although it is not possible to trace out the course of events in detail, it is fairly certain that the process ends by fission into two stars which at first describe orbits of small eccentricity about one another. After this point the story is carried on by tidal friction. As soon as the body divides, the dissipative forces which have so far produced secular instability become identical with those which now produce tidal friction, while the oscillations of the star become resolved into the oscillations of the co-ordinates of the two constituents which, in the way already described, constitute the description of an elliptic orbit.

Thus the beginning and end of the process of fission are known in some detail. At the beginning there is a rotation and an oscillation, and the periods of the two are different and in general incommensurable. At the end there is a rotation (or rather by now a revolution) and an oscillation, and the periods of the two are the same, except for a slight difference arising from the fact that the two constituents, not being perfectly spherical, do not attract according to the exact inverse square law. Except for this insignificant correction, we may conclude that during the process of fission the periods of oscillation and of rotation gradually approximate to one another, and that they have become identical by the time the fission is complete.

Thus at the beginning of fission the light curve would exhibit two (or possibly more) superposed and generally incommensurable periods. There would be no regularly recurring maxima, and in its general characteristics the light curve would resemble that of the "irregular" variables. Before the end of the fissional process there would be only one period, that of the star's rotation, and the light curve would show a definite periodicity with regularly recurring maxima. In this respect the light curve would be similar to that of the Cepheids and of the majority of the long-period variables. After the fission is complete this period persists as that of the star's revolution and is gradually modified in accordance with the theory of tidal friction. We are thus led to conjecture that the following scheme may represent the process of fission :—

Shape of Star.	Class of Star.	Light Curve.	Spectrum.	Range.
Ellipsoid.	Unobserved	..	Dark.	..
Pear-shaped.	Irreg. variable.	Periods superposed.	M, N.	Up to 6 m.
,,	Long-period variable.	Traces of irregularity.	M.	
,,	Cepheid.	Regular periodicity.	M to A.	0·4 to 1·0 m.
Two ellipsoids.	Early spectroscopic and eclipsing binaries.	Orbital periodicity.	A, B.	Up to 0·7 m.

The last column but one shows the progress of spectral type which, as we have seen, must necessarily accompany the progress of fission, no matter how rapidly the process of fission takes place.

14. It will be seen that this scheme requires that the mass of the long-period variable or of the Cepheid variable shall be equal to the sum of the masses of the two constituents of the binary which it will ultimately form by fission. That is to say, the mass of certain stars of types M, K, G, F must be supposed to be about double that of certain other stars of types B and A. Some astronomers may feel that this is a grave objection ; there is a general belief, supported to some extent by the recent researches of Seares, that even in the giant stars the early type stars are the most massive. As Shapley has pointed out, however, general conclusions of this kind are often better formed and better tested from a study of distant star-clusters than from a study of the galactic system, since in studying the star-clusters the errors arising from uncertainties of parallax and the disturbances resulting from selection by nearness are avoided. Shapley's studies of the star-clusters seem to indicate quite definitely that in them at least the M stars are brighter than the B stars, whence, *à fortiori*, the M stars must be more massive than the B stars.

Newly formed binaries exhibit a large range in mass and density, whence we must suppose that stars in which the process of fission is just commencing must exhibit a similar range. Thus no single star must be expected to cover the whole range of density from the most diffuse of irregular variables, such as α Orionis, up to the density of the shortest period eclipsing binaries. A certain amount of control is provided by the principle of conservation of angular momentum, so that it can be shown that α Orionis can never break up into a binary of very short period, while RR Lyræ cannot, when fission commenced, have been a star of very low density. The full discussion of this question must, however, be reserved for a later paper.

15. From what has been said, it is clear that when a star which is in process of breaking up shows a definite periodicity, the periodicity must be that of the star's rotation. Thus according to our scheme a Cepheid of twenty days' period must be rotating in twenty days.

This provides a check on our theory, since in any star in which fission is in progress there is a simple relation between a star's density and its period of rotation. Let us consider first the ideal case of an incompressible mass. When instability first sets in, the density ρ and the velocity of rotation ω are connected by the relation

$$\frac{\omega^2}{2\pi\gamma\rho}=0{\cdot}14200 \quad . \quad . \quad . \quad . \quad (8)$$

When the star has broken into two equal masses which are assumed for convenience to be spherical and rotating in contact, the relation is

$$\frac{\omega^2}{2\pi\gamma\rho}=0{\cdot}16667 \quad . \quad . \quad . \quad . \quad (9)$$

Inequality in the two constituent masses does not greatly alter this

figure, and neither does homogeneity of the matter provided that ρ is taken to denote the mean density.

Thus when a star shows a quite regular period ω^2 and ρ ought to be connected by an equation of the general type of (8) and (9). Probably equation (9) provides a good enough approximation. Denoting the period in days by P, this can be written in the form

$$\rho=\frac{0{\cdot}075}{P^2} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In the following table the average period for Cepheids of different type are those estimated by Shapley* ; the fourth column gives the density of the Cepheid as calculated from relation (10). The last two columns give the average densities of stars of the same spectral type as estimated respectively by Seares and Russell.

Star.	Type.	Average Period.	ρ (calc.).	ρ (Seares).	ρ (Russell).
Long period	M	350	·0000006	·0000006	..
Cepheid	M	33	·00007	..	..
„	K	18	·0002	·00001	·0005
„	G	11	·0006	·00002	·0040
„	F	6	·0021	·0004	..
„	A	0·4	·47	·008	·14

As Shapley has pointed out, a relation of the type of (10), but with a different numerical factor, would hold on the pulsation theory also. The calculated densities agree with observation about equally well on either hypothesis ; both hypotheses survive this test, but the pulsation theory appears to the present writer to be ruled out by the objections previously stated.

16. A further test is provided by calculating ρ from our theoretical equation (1). If a is the radius of the star, assumed for simplicity to be spherical, we have $M=\frac{4}{3}\pi\rho a^3$ and $E\propto T_e{}^4a^2$. In terms of logarithms these equations and equation (10) become

$$\log M=\log\rho+3\log a+\text{cons.} \quad . \quad . \quad . \quad (11)$$

$$-0{\cdot}4m=4\log T_e+2\log a+\text{cons.} \quad . \quad . \quad . \quad (12)$$

$$\log P+2\log\rho=\text{cons.} \quad . \quad . \quad . \quad . \quad (13)$$

Let us write the theoretical equation (1) in its approximate form (*cf.* equation (3)).

$$c_1\log T_e+c_2\log M+c_3m=\text{cons.} \quad . \quad . \quad . \quad (14)$$

We have already noticed that over the range of mass $5\odot$ to $20\odot$, which is approximately that of the Cepheid variables, the quantity θ varies only slowly with M, so that this equation ought to give a good approximation for the Cepheid variables.

Eliminating a, T_e, and M from equations (11) to (14), we obtain

$$(1{\cdot}2c_2-2c_3)m+4c_2\log P+(12c_2-2c_1)\log T_e=\text{cons.} \quad . \quad (15)$$

* *Astrophys. Journ.*, **44** (1916), 273.

It is difficult to determine the constants c_1, c_2, c_3 in equation (14) with any accuracy, but all investigations agree in showing that c_2 is quite large in comparison with c_1 and c_3. If, as a first rough approximation, we neglect the ratios c_1/c_2 and c_3/c_2 altogether, equation (15) becomes

$$0{\cdot}3m+\log P+3\log T_e=\text{cons.} \quad . \quad . \quad . \quad (16)$$

If we treat T_e as a constant, thereby limiting ourselves to a small range of spectral type, equation (16) becomes

$$\log P+0{\cdot}3m=\text{cons.} \quad . \quad . \quad . \quad . \quad (17)$$

This expresses the period-luminosity law. The curve given by Shapley to express this law has for its equation in the region for stars brighter than abs. mag.-2, approximately

$$\log P+0{\cdot}29m=\text{cons.} \quad . \quad . \quad . \quad (18)$$

which is in satisfactory agreement with (17).

Further approximations to the period-luminosity law present a more difficult problem which is reserved for another paper.

17. At any instant let E denote the total rate of emission of energy of a variable star, let A be the total area of its surface, and let θA be the area visible to a terrestrial observer. Disregarding the "darkening" effect at the limb, the star's brightness will appear to a terrestrial observer to be proportional to θE, so that its estimated absolute magnitude will be of the form

$$-0{\cdot}4m=\log\theta+\log E+\text{cons.} \quad . \quad . \quad . \quad (19)$$

In this equation both θ and E will vary periodically, the period of θ being either that of the star's rotation or else half of this period, while the period of E will be that of the star's oscillations. We have seen that in the earlier stages of fission these two periods will in general be unequal and incommensurable, so that the light curve represented by equation (19) will not exhibit regularly spaced maxima and the star will be classified as an "irregular" variable. At a later stage the periods of θ and E will become equal and the light curve will show regular periodicity.

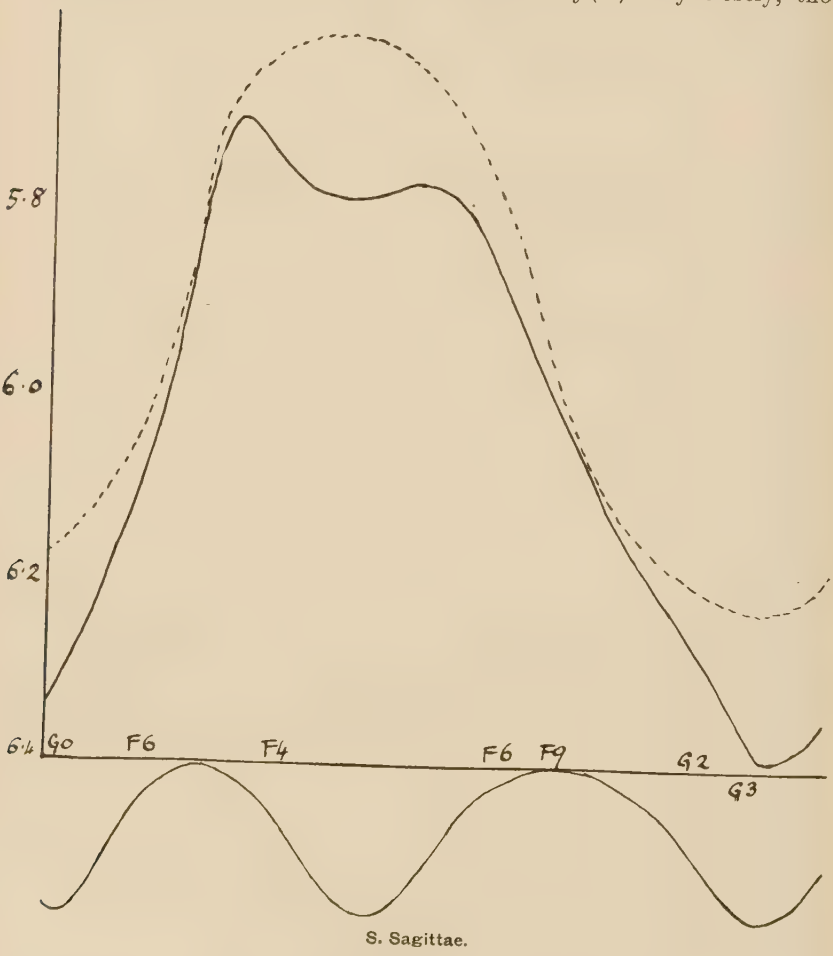
When this latter stage is reached, let the frequency of the star's oscillation be $n/2\pi$, so that log E will be of the form $f(nt)$, where $f(nt)$ is periodic in nt with a period 2π. Then log θ will be of the form $\phi(2nt)$ if the rotating body is of such a shape that θ repeats itself twice in a revolution: otherwise it will be of the form $\Phi(nt)$. The rotational terms will, of course, differ in phase to any extent from the term $f(nt)$ which arises from oscillations of the star's mass. If for simplicity we suppose $\phi(2nt)$ and $\Phi(nt)$ to be simple cosine terms with constants added, equation (19) will assume the alternative forms

$$m=m_0+A\cos(2nt-e)+Bf(nt) \quad . \quad . \quad . \quad (20)$$

$$m=m_0+A'\cos(nt-e')+B'f(nt) \quad . \quad . \quad . \quad (21)$$

18. An analysis of the light curves of Cepheid variables suggests that these can be expressed by equations of these forms (or possibly by equation (20) alone) where the graphs of $f(nt)$, as a function of nt,

are very similar for all Cepheids, the observed differences in the light curves arising from variations in the phase e and in the ratio of the coefficients B/A or B′/A′. Further, the spectral variations as recorded by Shapley * appear to follow the value of $f(nt)$ very closely, the



S. Sagittae.

FIG. 2.

maximum of $f(nt)$ coinciding with the earliest type of spectrum attained by the star, and *vice versa*. This is, of course, precisely what ought to be expected if the term $f(nt)$ represents the effect of the physical oscillation of the star, while the cosine terms depend on the star's orientation relative to our line of sight. Fig. 2 shows the light curve of the Cepheid variable S Sagittæ † analysed in the way suggested.

* *Astrophys. Journ.*, **44** (1916), 273.

† *Cf.* Shapley, *loc. cit.*, p. 288.

The continuous curve above the axis is the observed light curve as given by Gore and Luizet, the dotted curve is the suggested graph of $f(nt)$, and the entries along the time-axis denote the spectral type as recorded by Shapley. In passing, we may notice how entirely impossible it is to interpret a light curve of this shape in terms of a pure pulsation hypothesis.

According to this view of Cepheid variation, the Cepheids ought to change continuously into long-period variables at one end and into eclipsing or spectroscopic binaries at the other end. There ought to be no clearly defined line of demarcation between Cepheids and binaries, and when the attempt to interpret the light and velocity curves in terms of the orbital motion of a binary leads to satisfactory and consistent results, there would seem to be no reason why this interpretation should not be adopted. When this interpretation fails, it seems natural to suppose that the process of fission is not yet complete and to attempt an explanation in terms of the oscillations of a rotating body, after the manner suggested in equations (20) and (21).

The constituents of a fully formed binary cannot legitimately be treated as particles, so that their mutual attraction, as has already been remarked, will not be precisely mm'/r^2, and the orbit will not be precisely elliptical. To a first approximation the correction to be applied to the ordinary Keplerian motion will consist of a rotation of the apsidal line of the orbit and a consequent slow variation in the light curve. The case of a binary in which the constituents are not yet fully separated is in general similar, but the irregularity is now more pronounced. There is a rotation of the line passing through the longest diameter of the mass when at its greatest elongation, the line which ultimately will develop into the apsidal line of the orbit, and this may be represented analytically by slow changes of the phases e, e' entering into equations (20) and (21). To a terrestrial observer it will appear that the light curve is gradually changing its character, and he will further observe fluctuations in the periods between successive maxima. Prolonged observation ought to suggest a cyclical period of the form

$$P=P_0+Q\cos \alpha t$$

where $2\pi/\alpha$ is the period of rotation of the embryo apsidal-line.

19. Thanks to the valuable work of Phillips, the discussion of the light curves of long-period variables is a much simpler matter. Phillips has analysed the light curve, as far as the third harmonic term, in the form *

$$m=P_0+P_1\cos(\theta-\pi)+P_2\cos(2\theta-\phi_2)+P_3\cos(3\theta-\phi_3) \quad . \quad (22)$$

He finds that all the stars fall into two groups, the first being distinguished by the fact that ϕ_3 has the approximately constant value $202^\circ\cdot1$, while the values of ϕ_2 and ϕ_3 in the second group are connected approximately by a linear relation which Phillips expresses as

$$\phi_3=1\cdot67\phi_2-126^\circ\cdot8 \quad . \quad . \quad . \quad . \quad (23)$$

* Presidential Address to the B.A.A., 1916 Oct. 25.

These values for ϕ_3 are, of course, only approximate, and I find that the value of ϕ_3 for the second group can almost as well be expressed by

$$\phi_3 = 1{\cdot}5\phi_2 - 107^\circ{\cdot}5 \quad . \quad . \quad . \quad . \quad (24)$$

This change involves the transfer of a few stars from Group II. to Group I.; it has the advantage, as we shall see, that it permits of a physical interpretation being given to the two groups.

Phillips finds for the coefficients P_1, P_2, P_3 values which, to a rough approximation, are in the same ratio for all the stars of the same group. Using the mean values for these coefficients, the light curves of the two groups are found to be

$$\text{(Group I.)} \quad m = 2{\cdot}03\cos(nt-\pi) + 0{\cdot}23\cos(2nt-\phi_2) + 0{\cdot}11\cos(3nt-202^\circ{\cdot}1) \quad (25)$$

$$\text{(Group II.)} \quad m = 2{\cdot}41\cos(nt-\pi) + 0{\cdot}44\cos(2nt-\psi_2) + 0{\cdot}18\cos(3nt-1{\cdot}5\psi_2+107^\circ{\cdot}5) \quad (26)$$

in which the phase ψ_3 in Group II. has been replaced by its value as given by equation (24).

If we refer the light curves of Group II. to new origins by the change of phase involved in writing $nt+0{\cdot}5\psi_2-103^\circ{\cdot}2$ for nt, equation (26) becomes

$$\text{(Group II.)} \quad m = 2{\cdot}41\cos(nt+0{\cdot}5\psi_2-283^\circ{\cdot}2) + 0{\cdot}44\cos(2nt-206^\circ{\cdot}4) + 0{\cdot}18\cos(3nt-202^\circ{\cdot}1) \quad (27)$$

Let us now put

$$f(nt) = 3{\cdot}32\cos(nt-\pi) + 0{\cdot}44\cos(2nt-206^\circ{\cdot}4) + 0{\cdot}18\cos(3nt-202^\circ{\cdot}1),$$

and we find that equations (25) and (27) assume the forms

$$\text{(Group I.)} \quad m = \mathrm{A}\cos(2nt-e) + \tfrac{1}{18}f(nt)$$
$$\text{(Group II.)} \quad m = \mathrm{A}'\cos(nt-e') + f(nt)$$

which are precisely identical with our theoretical equations (20) and (21).

This last result suggests strongly that the long-period variables may admit of explanation in terms of the theory which led to equations (20) and (21)—in other words, that they may be interpreted as stars in the process of fission. The details of course remain to be worked out; at present the suggestion is merely put forward on broad lines for consideration and, it is hoped, for discussion.

20. One interesting possibility may be noticed in conclusion. The majority of M-type long-period variables show a single definite period, which, on our theory, suggests that they are well advanced in the process of fission. This, together with the numerical values we calculated in § 5, combine to suggest that normally fission must start while a star is still non-luminous.

This indicates a possible way of clearing up certain difficulties which beset the general question of the birth of stars in spiral nebulæ.

The total luminosity of a nebula has always seemed to be far too small compared with that of the stars which calculation shows ought to be

born out of it. So long as we are concerned only with visible radiation this is certainly the case, but it now appears possible that the great bulk of the radiation both from the nebulæ and the stars may be invisible. Just as fission raises the temperature of a star's surface, so the break-up of a nebula into stars may raise the temperature of the surface of the stellar matter involved. To put the matter in an extreme way, it is conceivable that the generation of energy in the matter of a nebula may fail to raise it to a temperature at which it emits any radiation capable of detection while it is in nebular form. When the matter has divided into stars the same total generation of energy may raise these stars to somewhere near to the limit of visibility, and further fission into binaries may raise them to about B-type. This would give an immediate explanation of Seares' observation that the outer parts of spiral arms are bluer than the inner parts.

Calculation * indicates that fission ought to commence in stars born out of spirals when they reach densities of the order of 10^{-6} or lower, and at this density a star of ordinary mass would be in a state of complete darkness. If stars of this low density were of such high temperature that their substance was fully ionised there would be some difficulty in understanding the process of fission, for fission requires more uniformity of density throughout the mass than we should expect in a mass of uniform substance at a density of the order of 10^{-6}. It is, however, possible that we have to deal with a mass whose outer parts consist of unbroken molecules and in which the ionisation steadily increases as we move towards the centre. If so, the difficulty would, to a large extent, disappear. A great deal more of mathematical analysis appears to be required before a final opinion can be expressed on this and on many similar questions.

The Emission of Hydrogen and Helium from a Star by Radiation-Pressure, and its Effect in the Ultra-violet Continuous Spectrum. By Martin C. Johnson.

1. In any interpretation of the connection between planetary and diffuse nebulæ and the stars which they surround,† it becomes desirable to know to what extent atoms of any element can be driven out from the central star. For instance, the emission of corpuscular rays from a very hot star has more than once been suggested as a means of excitation of the spectrum of a surrounding nebulosity. The problem of the escape of atoms from an atmosphere in gravitational equilibrium has been worked out rigorously by Milne.‡ The result of his work is as follows :—

(*a*) Atoms of hydrogen, and *a fortiori* all heavier atoms, cannot

* *Cosmogony and Stellar Dynamics*, § 255.

† *Lick Obs. Pub.*, **13**; Hubble, *Ap. J.*, **56**, 162 and 400, 1922; Russell, *Observatory*, **44**, 72, 1921.

‡ *Camb. Phil. Soc. Trans.*, **26**, 512, 1923.

escape even from a giant star with very low value of the acceleration of gravity at its surface.

(*b*) The only particles which could escape are electrons, and the resulting positive charge which would spread over the ionised surface layer of the star would soon prevent any but slight loss of these.

(*c*) These conclusions are only valid if radiation-pressure can be neglected.

The present paper is an attempt to calculate the escape of certain atoms from a certain type of chromosphere which is supported by radiation-pressure. The method is based on the now generally accepted view that radiation-pressure on an individual atom is just the gain of momentum associated with the absorption of energy by electrons in various quantum orbits ; use is made of recently determined values of atomic absorption coefficients, and of the Saha-Fowler-Milne equations for the distribution of atoms in states capable of absorbing various monochromatic energies. These energies are calculated on Planck's equation of black radiation. The conclusions arrived at are applied to the interpretation of the continuous spectrum seen beyond the Balmer limit in certain stars and nebulæ.

2. The type of atmosphere to be considered has been made familiar by Milne.* In its lower parts a Maxwellian distribution of velocities still holds, but in its upper portions the density is so low that gravitation is effectively the only force balancing the selective absorption of momentum from that fraction of photospheric radiation which has not been absorbed in a reversing layer at the base of the atmosphere. Milne points out that for certain atoms in stars with certain gravitational properties and certain intensities of absorption lines, such a rarefied chromosphere will not be supported unless the star has a certain minimum effective temperature : since the present problem was suggested particularly by the case of a planetary nebula, we can assume that a chromosphere is formed and is either in equilibrium, of the type discussed by Milne, or else loses atoms outwards. Let us take an effective temperature of 40,000°, which is certainly not more than is implied by the spectral energy distribution curves of the nuclei of planetary nebulæ, and is of the order of the hottest stars whose reversing layer temperatures can be calculated from thermodynamic equilibrium. In using the calculated intensity of black radiation at this temperature, it must be remembered that departures from black body distribution, which may occur owing to extensive scattering, will imply that the actual temperature is higher than that deduced from spectral energy curves.†

Consider the possibility of the emission from such a chromosphere, firstly of free electrons, then of those atoms which will, apart from radiation-pressure, be found at the greatest heights. Milne has given reasons‡ why the lightest elements, hydrogen and helium, do not appear higher in a star's chromosphere than the heavier calcium : but in a star of the temperature we are to consider, the intensity of the total radiation is concentrated in shorter wave-lengths than in the sun, and the lack of

* *M.N.*, **84**, 354, 1924, and **85**, 111, 1924.
† Milne, *Phil. Trans. Roy. Soc.*, A, 223, 237, 1922.
‡ *M.N.*, **84**, 354.

ultra-violet resonance radiation which keeps them low in the sun will not here be felt. Except for the fact that the resonance frequencies occur outside the range of appreciable solar radiation, H and He would be found higher than any other elements; accordingly these two will be discussed in this paper, for their possible emission from the star.

3. Consider first the case of free electrons. The types of obstruction to radiation in the visual region which can be exerted by free electrons are not unanimously agreed on; * if we adopt the only experimentally known mechanism we must use the scattering coefficient

$$s/\rho = 1{\cdot}4 \times 10^{-2},$$

where ρ is the density of a gas containing approximately as many free electrons as atoms. A convenient way of calculating the energy scattered by a single free electron is provided by the concept of a "Cross-section for absorption," due to Eddington. If the intensity of radiation crossing unit area in a second is considered, the proportion scattered by a single electron or absorbed by a single atom, can be found from a quantity of the physical dimensions of an area and equal to

$$m\frac{s}{\rho} \quad \text{or} \quad m\frac{\kappa}{\rho},$$

where s and κ are scattering or absorption coefficients and m is the mass of a single particle.

The amount scattered by a free electron will therefore be

$$m\frac{s}{\rho}\pi \mathrm{F},$$

where F is the intensity of black body radiation at temperature T, given in terms of Stefan's constant σ by the relation

$$\mathrm{F} = \int_0^\infty \mathrm{F}_\lambda d\lambda = \sigma \mathrm{T}^4/\pi.$$

The momentum per second associated with the energy scattered will be

$$\frac{1}{c} m \frac{s}{\rho} \pi \mathrm{F}.$$

On inserting $c = 3 \times 10^{10}$, $m = 9 \times 10^{-28}$, $s/\rho = 1{\cdot}4 \times 10^{-2}$, $\sigma = 5{\cdot}7 \times 10^{-5}$, $\mathrm{T} = 40{,}000°$, this is found to be

$$6 \times 10^{-26}.$$

Now this is the maximum possible momentum which could be gained by the electron if all the momentum lost to the forward beam of radiation were transferred to it. Actually the momentum gained will be between zero and this value, according to the distribution of the scattered radiation.

Now consider the opposing gain of momentum from gravity. A star of this temperature will have a large mass, and hence a large value of g,

* *M.N.*, **85**, 56.

the gravitational acceleration at the surface. But the O-type stars are all of very high luminosity, the anomalous faintness of the planetary nebulæ being for other reasons physically impossible as intrinsic phenomena and accepted as due to some form of screening. If M is the mass, R the radius, L the absolute luminosity, and T the effective temperature of a star,

$$g \propto M/R^2,$$
$$L \propto T^4R^2,$$
$$\therefore\ g \propto MT^4/L.$$

Hence extreme values of the gravitational attraction need not be expected; to take a single value we will adopt that found by Milne * for Plaskett's extremely massive star.

The acceleration upward of the electron due to radiation-pressure will be

$$\text{momentum per second/mass.}$$

From the value previously calculated on the scattering theory, this is equal to

$$6{\cdot}6\times10^{1}\text{ cm. per sec.}^2,$$

whereas the downward acceleration at the surface of Plaskett's star is

$$5{\cdot}6\times10^{3}\text{ cm. per sec.}^2.$$

We conclude that even at this high temperature the maximum momentum which a free electron could gain from radiation is only of the order of 10^{-2} times the momentum which it will gain gravitationally in the same time. Hence, unless free electrons have other ways of obstructing radiation it seems unlikely that β-rays can be emitted from a star by radiation-pressure.

4. Now the absorption coefficients calculated by Milne † and measured by R. W. Wood are solely concerned with the reaction between radiation of a definite wave-length and an atom in a definite quantum state, whereas the scattering coefficient for free electrons is not selective either with respect to the state of the electron or the wave-length of the radiation. The energy available per second, on the average, for true atomic absorption can therefore be expressed by

$$F_\lambda = r\tau \frac{2hc^2}{\lambda^5\left(e^{\frac{hc}{\lambda\kappa T}}-1\right)}d\lambda.$$

Here $2hc^2/\lambda^5(e^{hc/\lambda\kappa T}-1)$ is Planck's expression for the distribution of energy in the spectrum of black radiation, $d\lambda$ is the width of the narrow range of wave-lengths which the atom concerned is able to absorb, r is the residual intensity left to that range by reversing layer absorption before the atom is reached, and τ is the fraction of its life that the atom spends in the suitable quantum state for absorption of this particular radiation.

Neglecting for the moment the question of τ, which we consider in detail in section 5, we will assume a line width of 2 A.U., corresponding

* *M.N.*, **84**, 360. † *Phil. Mag.*, **47**, 209, 1924.

to $d\lambda = 2 \times 10^{-8}$ cm. A reasonable value for the fraction r may be derived from the following considerations. Studies of the spectra of the nuclei of planetary nebulæ may be found in the *Lick Observatory Publications*, vol. xiii., and the papers of Hubble.* They agree in stating that the nucleus shows a predominantly continuous spectrum, sometimes with Wolf-Rayet bands, but these not contrasting strongly with the continuous background, and sometimes with faint H and He^+ absorption lines. Hence the narrow strips of spectrum we are here concerned with are either slightly brighter than the surrounding continuous radiation or their brightness is suppressed to the extent of slight absorption. The continuous radiation notoriously extends far into the ultra-violet, the maximum being beyond the visible limit. Hence the fraction, which in a star with very strong absorption lines would reduce the energy calculated from black body emission, is not important in the cases with which we are concerned, and will not differ greatly from unity.

The familiar absorption lines of the H atom are the Balmer series, but are not derived from the normal state of the atom, in which the Lyman series would be absorbed. Helium, similarly, in its normal state would absorb a series some of whose lines have been measured in the far ultra-violet, and which probably converges not far from λ500. But whereas ionised hydrogen cannot absorb at all until it captures an electron, there are three known series of radiations associated with ionised helium, as well as those which include the singlets and doublets of the neutral but excited atom.

Inserting in the Planck equation the values $h = 6{\cdot}55 \times 10^{-27}$ erg sec., $c = 3 \times 10^{10}$ cm. per sec., $\kappa = 1{\cdot}37 \times 10^{-16}$, and taking $d\lambda = 2 \times 10^{-8}$ cm. and $r = 1$, the energy available for absorption by H and He in the several quantum states is calculated from the limits of the series concerned. Since the largest error is that involved in τ, a high order of accuracy is not required.

Atom.	Wave Number at Limit of Series which can be absorbed.	Series.	F_λ. Energy available for Absorption at Limit, in ergs per cm.^{2} per sec.	
H	$N/1^2 = 109678$	Lyman	7×10^{10}	$\times\tau$
H	$N/2^2 = 27420$	Balmer	2×10^{9}	$\times\tau$
H^+	Cannot absorb.			
He	Lines for normal atom at about λ500	..	5×10^{10}	$\times\tau$
He	$1P = 27175$	(singlets)	2×10^{9}	$\times\tau$
He	$1\pi = 27224$	(doublets)	2×10^{9}	$\times\tau$
He^+	$4N/2^2 = 109723$	..	7×10^{10}	$\times\tau$
He^+	$4N/3^2 = 48766$	"4686"	13×10^{9}	$\times\tau$
He^+	$4N/4^2 = 27431$	Pickering	2×10^{9}	$\times\tau$
He^{++}	Cannot absorb.			

5. Since it is evident that at 40,000° both the elements we are considering must be highly ionised, the time during which hydrogen can absorb any of these energies will be very small indeed, and even for

* *Ap. J.*, **56**, 162 and 400.

helium it will be limited by second ionisation. Hence the investigation of τ becomes very important.

The only mathematical relations available for finding τ are based on thermodynamic equilibrium. Now although thermodynamic equilibrium may be expected to hold in a star's reversing layer with only small exceptions of the kind discussed in a previous note,* it is clearly not a valid assumption for high chromospheric layers. But at great heights the ionisations may be taken as almost entirely radiative,† and these will be approximately as many as would characterise thermodynamic equilibrium in a gas at 40,000°, even though at these heights the state of the atmosphere does not justify our assuming a Maxwellian distribution of velocities corresponding to 40,000°. Hence, if we can find values for τ for the highest layers which are in thermodynamic equilibrium, these may be taken as approximately correct for the upper layers from which atoms may be driven out.

If n_1, n_2 be the relative concentrations of atoms in two different quantum states at any one time, and τ_1, τ_2 are the times an atom spends in these particular states,

$$\tau_1/\tau_2 = n_1/n_2.$$

But where more than two states are involved—as in the present case—a more complicated relation is required. We will consider a case of three components.

Let n_1, n_2, n_i represent the fractional concentrations of atoms in the normal, excited, and singly ionised states. Then to relate to these the corresponding values of τ, adopt the probability coefficients first used by Einstein and familiarised in this country by Milne.‡ Some of these are atomic constants, and others are proportional to the appropriate radiation intensities. Let

$$\mathrm{A(I)}_{1\to 2}dt, \quad \mathrm{A(I)}_{1\to i}dt, \quad \mathrm{A(I)}_{2\to i}dt$$

represent the probability coefficients of transference from state 1 to state 2, etc. These probabilities are all functions of the appropriate intensity I. Let

$$\mathrm{A}_{2\to 1}dt, \quad \mathrm{A}_{i\to 1}dt, \quad \mathrm{A}_{i\to 2}dt$$

represent the coefficients of the converse transference from state 2 to state 1, etc. Then, if we assume that each transition is balanced separately by its converse,

$$\begin{aligned} n_1\mathrm{A(I)}_{1\to 2}dt &= n_2\mathrm{A}_{2\to 1}dt \\ n_1\mathrm{A(I)}_{1\to i}dt &= n_i\mathrm{A}_{i\to 1}dt \\ n_2\mathrm{A(I)}_{2\to i}dt &= n_i\mathrm{A}_{i\to 2}dt. \end{aligned}$$

The probability of leaving state 1 is proportional to $(\mathrm{A(I)}_{1\to 2}+\mathrm{A(I)}_{1\to i})$, and since τ_1, τ_2, τ_i are inversely proportional to the probabilities of leaving each state,

$$\begin{aligned} \tau_1 &= 1/\{\mathrm{A(I)}_{1\to 2}+\mathrm{A(I)}_{1\to i}\} \\ \tau_2 &= 1/\{\mathrm{A}_{2\to 1} \quad +\mathrm{A(I)}_{2\to i}\} \\ \tau_i &= 1/\{\mathrm{A}_{i\to 1} \quad +\mathrm{A}_{i\to 2}\}. \end{aligned} \qquad . \quad . \quad . \quad (1)$$

* *M.N.*, **84**, 516.

† See Fowler, *Phil. Mag.*, **47**, 275, 1924.

‡ *Phil. Mag.*, **47**, 209.

Hence, in terms of the absorption probability coefficients,

$$\tau_2 = n_2/\{n_1 \mathrm{A(I)}_{1\to 2} + n_2 \mathrm{A(I)}_{2\to i}\} \quad . \quad . \quad . \quad (2)$$

$$\tau_i = n_i/\{n_1 \mathrm{A(I)}_{1\to i} + n_2 \mathrm{A(I)}_{2\to i}\} \quad . \quad . \quad . \quad (3)$$

τ_1 is already expressed in terms of outward probabilities only.

First consider the methods based on thermodynamic equilibrium, by which n_1, n_2, n_i can be calculated for H and He.

In the Saha-Fowler-Milne equation *

$$\log \frac{x}{1-x} \mathrm{P}_e = \frac{-\chi_1}{\kappa \mathrm{T}} + \tfrac{5}{2} \log \mathrm{T} + \log \frac{(2\pi m)^{\frac{3}{2}} \kappa^{\frac{5}{2}} \sigma}{h^3} - \log b(\mathrm{T}) \quad . \quad (4)$$

x is the fraction of atoms ionised, P_e is the partial pressure of electrons, χ_1 the ionisation potential, κ Boltzmann's constant, T the temperature, m the mass of an electron, h Planck's constant, σ the number of valency electrons in equivalent orbits, and b(T) is the partition function defined by q_1, q_2 . . ., the weights of the different quantum states. From this we can find the ionisation at 40,000°, and at a reversing layer pressure of 10^{-4} atmosphere ($=10^2$ c.g.s. units), which is accepted as a reasonable mean. We use the following numerical values :—

H	$b(\mathrm{T})=2$	$\sigma=1$	$\chi_1=13{\cdot}54$ volts
He	$b(\mathrm{T})=1$	$\sigma=2$	$\chi_1=24{\cdot}7$ volts
He^+	$b(\mathrm{T})=2$	$\sigma=1$	$\chi_1=54{\cdot}2$ volts.

We find then, from equation 4,

$$1-x(\mathrm{H}) = 10^{-7}$$
$$1-x(\mathrm{He}) = 0{\cdot}7\times 10^{-6}.$$

Since the neutral fraction is so small compared with the ionised fraction, we may apply the same equation to calculate the second stage ionisation of He, for which we find

$$1-x(\mathrm{He}^+) = 1{\cdot}3\times 10^{-2}.$$

For the distribution of the neutral fraction in different states the same theory gives the fraction of neutral atoms in an r-state as

$$f_r = \frac{q_r e^{-\frac{(\chi_1-\chi_r)}{\kappa \mathrm{T}}}}{b(\mathrm{T})},$$

where χ_r is the energy of the r-level, and q_r is the corresponding weight, $(\chi_1-\chi_r)$ being a resonance potential. Now the states with which we are concerned are those in which the energies tabulated on p. 817 can be absorbed. For H in the Balmer state $\chi_1-\chi_r=10{\cdot}16$ volts, and $q_r=r(r+1)=6$. Hence

$$f_r(\mathrm{H}) = 0{\cdot}15.$$

For He the energies for the singlet and the doublet series we have

* *M.N.*, **83**, 407, 1923.

tabulated are not different in the first significant figure, and we may take as a mean value $\chi_1-\chi_r=21{\cdot}2$ volts, and $q_r=2$. Hence

$$f_r(\mathrm{He})=4\times10^{-3}.$$

Applying these results to the evaluation of the concentrations, we find for H

$$n_1=0{\cdot}85\times10^{-7}$$
$$n_2=0{\cdot}15\times10^{-7},$$

while

$$n_1+n_2+n_i=1.$$

If we take fractions of the neutral atoms of He we find

$$n_1=0{\cdot}7\times10^{-6}$$
$$n_2=0{\cdot}3\times10^{-8},$$

but in this case

$$n_1+n_2+n_i\fallingdotseq1{\cdot}3\times10^{-2}.$$

It remains to determine the probability coefficients $\mathrm{A(I)}_{1\to2}$, etc., which together with n_1, n_2, etc., will give τ_1, τ_2, etc.

These coefficients have been related by Milne * to the monochromatic mass absorption coefficient κ_ν and the mass of an atom m by the formula

$$\mathrm{A(I)}_{1\to2}=\frac{\kappa_\nu+\pi m}{h\nu}.$$

Monochromatic mass absorption coefficients have been found from several sources, experimental and theoretical, and for the visual region have values between 10^8 and 10^9. We will take a mean value of 5×10^8 and insert $m(\mathrm{H})=1{\cdot}64\times10^{-24}$, $m(\mathrm{He})=4m(\mathrm{H})$. If then we assume that there is no great difference in the intrinsic probabilities connecting each absorption coefficient with the several frequencies which can be absorbed, we can calculate the probability coefficients required, since the radiation intensities implicit in $\mathrm{A(I)}_{1\to2}$, etc., can be obtained from the series limits. For hydrogen :

$$\mathrm{A(I)}_{1\to2}=\ 6\times10^{-4}$$
$$\mathrm{A(I)}_{1\to i}=\ 5\times10^{-4}$$
$$\mathrm{A(I)}_{2\to i}=20\times10^{-4}.$$

For helium :

$$\mathrm{A(I)}_{1\to2}=13\times10^{-4}$$
$$\mathrm{A(I)}_{1\to i}=11\times10^{-4}$$
$$\mathrm{A(I)}_{2\to i}=80\times10^{-4}.$$

The values found for n_1, n_2, n_i and the probability coefficients can now be inserted in equations 1, 2, 3, whence it is found that the fractions of a second during which absorption is possible are as follows. For hydrogen :

$$\tau_1=6{\cdot}0\times10^{-8}$$
$$\tau_2=1{\cdot}4\times10^{-8}$$
$$\tau_i\fallingdotseq1.$$

* *Phil. Mag., loc. cit.*, 217 and 227.

For helium :

$$\tau_1 = 4{\cdot}0 \times 10^{-9}$$
$$\tau_2 = 3{\cdot}0 \times 10^{-11}$$
$$\tau_i \fallingdotseq 1{\cdot}3 \times 10^{-2}$$
$$\tau_{ii} \fallingdotseq 1.$$

6. The energy which atoms of hydrogen and helium absorb in a second can now be calculated from the energies available at the series limits, tabulated on p. 817, multiplied by the times during which they are capable of these absorptions.

If $m\kappa/\rho$, the cross-section for absorption, be given the values which we have used already on p. 820, the momentum absorbed per atom per second will be

$$\frac{1}{c} m \frac{\kappa}{\rho} F_\lambda.$$

The acceleration of the atom upward due to radiation-pressure will be

momentum per sec./mass,

and can be directly compared with the gravitational acceleration, taken as before as equal to $5{\cdot}6 \times 10^3$ cm. per sec.^{2}.

From the table of energy and the values of τ we find that for a hydrogen atom Lyman absorption will contribute an acceleration of 71·0 cm. per sec.^{2}, and Balmer absorption will contribute 0·48 cm. per sec.^{2}. These absorptions are the only ones possible for an H atom, since when once ionised it has no other valency electron with which radiation can react. Hence, owing to the fact that at these temperatures all except a very small fraction of its life is spent in the ionised state, even the radiation-pressure from the hottest stars cannot make a great difference to the gravitational equilibrium of hydrogen.

For helium, conditions are different. Here the singly ionised atom can still absorb by the second valency electron, and it is not until double ionisation is complete that absorption, and, consequently, radiation-pressure, ceases. Although the majority of He atoms are doubly ionised at this temperature, the fraction singly ionised is enough to ensure large accelerations from radiation-pressure. The contributions from the normal and excited states are 3·4 cm. per sec.^{2} and 0·001 cm. per sec.^{2}, but from the singly ionised state an acceleration of from $4{\cdot}4 \times 10^5$ cm. per sec.^{2} to $1{\cdot}5 \times 10^7$ cm. per sec.^{2}, according to the distribution of the several He^+ orbits. Hence, without further analysis of the He^+ states, it is clear that the minimum value of the radiative acceleration of a helium atom will be 80 times, and the maximum value 2700 times, the gravitational acceleration.

7. The foregoing analysis has brought us to the conclusion that, while electrons and hydrogen atoms cannot be driven out from a stellar atmosphere by radiation-pressure against the pull of gravity, helium atoms will be driven out by a force 80--2700 times that of gravitation, until their sojourns in the He^{++} state are so long that no radiation can be absorbed. This conclusion has an application to the continuous spectrum seen in certain stars beyond the Balmer limit, which was first observed by Huggins and by Evershed, and of which

discussions have been published by Wright * and by Nicholson.† It is now a generally accepted view that as an absorption spectrum it is due to the photo-electric emission of electrons, and as an emission spectrum to the capture of electrons at high speed or to the disturbance communicated to an atom by an electron moving from one hyperbolic orbit to another hyperbolic orbit without actually settling into an elliptic orbit inside the atom. The detailed theory has been given by Sommerfeld.‡ If G is the limit of the spectral series associated with the final orbit (emission spectrum) or the initial orbit (absorption spectrum), and E is the kinetic energy of the free electron before capture or after ejection, then the frequency of the radiation emitted or absorbed is given by

$$\nu = G + E/h,$$

showing that the spectrum is continuous, since E is not quantised. It is important to notice the conditions under which the absorption and the emission cases of this spectrum occur. As an absorption spectrum it may be seen in ordinary A-type stars with strong hydrogen lines, but as an emission spectrum it seems to characterise regions of extremely low pressure ; it is seen in the solar chromosphere and prominences and in planetary and diffuse nebulæ and novæ, under conditions of pressure too low for Maxwellian distribution of velocities to be maintained ; that is to say, under just the pressure conditions of the chromosphere we have been considering in this paper. Wright found the velocity at which electrons would have to be captured to account for the observed emission spectrum, but only calculated it as a velocity of thermal agitation. But if the region is too rarefied for Maxwellian calculations, other sources must be looked for, for the electronic velocities.

In the atmosphere we have been studying there is a source of such velocities, which is associated with just that state of extreme rarefaction, and hence may reasonably be invoked to explain the spectrum which only occurs under nebular and similar conditions. Free electrons, we have found, are not subject to sufficient radiation-pressure to prevent their falling back into the star : now owing to the rarity of a collision at the very great distances to which helium is driven out, the electrons detached photo-electrically at these heights will have no chance of recombining until they reach a region of greater concentration of atoms, and hence will at once fall back towards the star. The enormous free paths at such heights will ensure that the electrons have reached a high velocity under gravitational attraction by the time they meet sufficient atoms to make capture probable, and the energy associated with this velocity of free fall will, when given up at capture into a Balmer hydrogen orbit or a Pickering helium orbit, provide the continuous emission spectrum seen. This explanation, if capable of providing quantitatively the energy which Wright calculated as thermal energy, would have the advantage of being peculiar to those regions in which the observed spectrum is peculiar.

* *Nature*, **109**, 810, 1922.
† *M.N.*, **85**, 253.
‡ *Atomic Structure*, 447.

8. Quantitatively we have to explain why, if the above explanation is correct, the continuous emission is stronger than the continuous absorption in those particular cases, *i.e.* why we can see more of captures into Balmer or Pickering orbits than photo-electric ionisations from Pickering orbits under these conditions.

The relation of emission spectrum to absorption spectrum is not the same as in the case of line radiation, and here we have to consider three possible reasons for the excess of emission spectrum :—

(*a*) More electrons are captured at the lower levels in the atmosphere than are detached at the highest levels to which helium can rise.

(*b*) Greater kinetic energy is given up at the encounter at the lower level than is given to the electron in ionisation.

(*c*) Greater energy is given up, but in more than one atomic encounter, each electron being responsible for several emissions of continuous spectrum.

There is no reason to expect a disturbance of statistical equilibrium in the direction required for (*a*) : the maximum number of captures, corresponding to the recombination of every electron detached at the higher levels, would provide an emission spectrum entirely masked by the absorption above. Given a very large velocity at capture, the emission spectrum might conceivably be strong at wave-lengths shorter than those seen in the photo-electric absorption spectrum, but this interpretation (*b*) is ruled out by the observed facts, that the emission is seen most strongly just beyond the Balmer limit and not further towards the violet than the corresponding absorption.

Let us turn to the third possibility (*c*). Since probability of capture has been shown to be inversely proportional to the square of the velocity of the electron,* the great energy attained by the free electron in its fall under gravity will not necessarily, or even probably, be given up in a single encounter ; hyperbolic-hyperbolic transitions with emission of radiation may reduce the kinetic energy until the speed is low enough to make capture more probable. If the hyperbolic and the capture encounters of a single free electron involve altogether a greater total loss of energy (gained gravitationally) than the kinetic energy with which the electron had been photo-electrically ejected at the higher level outside the atmosphere, the effect on the spectrum will be as if there were more than one capture to each ionisation, and the intensity of emission spectrum may be greater than the corresponding absorption.

To find whether this suggestion could account for the fact that we can see the emission in certain regions of great rarefaction, consider the energy which the free electron has to give up. To obtain a minimum value of its velocity, take the order of magnitude of the upper solar chromospheric density given by Fowler and Milne,† 10^{-13} atmosphere, 10^{-4} collisions per atom per second. This would allow of a free fall under gravity, which, if $g=6\times10^3$, would have a terminal velocity of 6×10^7 cm. per sec. The free paths and consequently the velocities reached in falling back, would generally be greater than this near the nucleus of a planetary nebula, since even in the sun matter seems to be driven to 8×10^5 miles (prominences) and the corona appears at 8×10^6

* Milne, *Phil. Mag., loc. cit.*, 226.

† *M.N.*, **83**, 418.

miles, while the above estimate of density related to the 14,000 kilometres to which calcium is observed to rise. Now the continuous emission has been seen as far as λ3340 in some nebulæ, for which the velocity of the electron before capture would have to be anything up to 3×10^7 cm. per sec. Hence, even if we take a small estimate of the kinetic energy which it could have gained gravitationally before it reached the capture level, and a large estimate of the kinetic energy it would have to give up to produce the spectrum required, the electron could contribute to the emission spectrum an amount of energy equivalent to that which would be absorbed in a corresponding equilibrium ionisation, and yet pass on and communicate the same amount again in capture to a second atom.

9. Finally, in all studies of the possible emission of charged particles from a star, the effect of electrostatic attraction and repulsion due to separated charges and to charge on the star as a whole must not be neglected.*

If positive ions escape entirely from a star, the resultant negative charge will spread itself over the ionised surface layers as over a conductor. If electrons escape, the charge which spreads over the surface of the star is positive. The effect in either case will be to check whatever escape of ions of one sign is taking place and to facilitate the escape of ions of opposite sign, so that the only escape which can go on indefinitely is one of equal numbers of positive and negative ions. Escape of atoms without radiation-pressure was shown by Milne to be negligible, and in the present theory of hydrogen and helium radiation-pressure is shown to cease when the appropriate degree of ionisation is complete, so that if we confined our attention to these light elements no complete escape of ions out into interstellar space would be possible. Such loss of atoms as does take place is more probably of elements higher in the periodic table, in spite of the fact that in gravitational equilibrium they would not tend to rise high in the chromosphere; for radiation-pressure on these atoms would not cease at double ionisation. This escape would also facilitate the non-radiative loss of electrons which is only held in check by electrostatic attraction.

The second electrostatic consideration which must be taken into account is the large-scale separation of charge accompanying the falling back of electrons to the recombination layer. Unless this were accompanied by the falling back of the H^+ and He^{++}, which are also unable to absorb, the field set up would check that free fall which, when the free paths are long enough, was suggested as a cause of the continuous emission spectrum. For this reason the extension of the theory to heavier elements is not to be called on to add to the intensity of the continuous emission spectrum; since their continued absorption beyond second ionisation will prevent their falling back as readily as the electrons, and they will encounter the above electrostatic difficulty avoided by H and He.

10. *Summary.*—The possibility of the emission of negative and positive ions from a star's atmosphere by radiation-pressure is investigated. The intensity of black radiation associated with the limits of the known

* See Milne, *Camb. Phil. Soc. Trans.*, *loc. cit.*

hydrogen and helium series is calculated for a very hot star under specified conditions. The equations of thermodynamic equilibrium are used to find approximately the distribution of H and He atoms in various quantum states, and hence to find during what proportion of the life of each atom it will be capable of absorbing the energy and momentum associated with the limits of the several series. It is concluded that electrons and H atoms will even in the nucleus of a planetary nebula be held back by gravitation against the maximum possible radiation-pressure. He atoms, on the other hand, will be subject to radiation-pressure up to 2700 times the attraction of gravity, and will therefore be ejected from the star until they reach a level when the concentration of atoms is too small for recombination and the He^{++} state becomes permanent, and, further absorption being impossible, they fall back.

The falling back of the electrons detached at the high levels will provide an energy from gravitation which will be given up in one or more hyperbolic collisions, and, finally, wholly transferred to atoms of H and He^{+} in capture at lower levels. Calculation of the energy available from this source suggests that these transfers will account for the occurrence, in these particularly rarefied regions, of the emission spectrum seen in place of an absorption spectrum in certain nebulæ beyond the Balmer limit.

Electrostatic considerations affecting the extension of the theory to heavier atoms are briefly discussed.

I am very grateful for the kindness of Professor E. A. Milne in giving me criticism and advice during the course of this investigation.

Physics Department, Birmingham University:
1925 *May.*

The Effect of Compton Scattering by Free Electrons in a Stellar Atmosphere.

By P. A. M. Dirac.

(*Communicated by Prof. E. A. Milne.*)

§ 1. *Introduction.*—When radiation is scattered by free electrons, on Compton's theory of scattering * there is an increase in its wave-length, the energy of each light-quantum being reduced by an amount equal to the energy of recoil of a scattering electron. Compton puts forward the suggestion † that this effect may account for the observed displacement of the lines in the solar spectrum to the red near the limb of the sun, it being assumed that the light from the limb is on the whole more scattered than that from the centre. Actually it is the continuous background on either side of the absorption line that is supposed to be shifted to the red, this causing a shift in the line itself.

To apply the Compton theory of scattering to a stellar atmosphere one must, however, take into consideration the velocity of thermal agitation of the electrons, and this may be expected to modify Compton's

* Compton, *Phys. Rev.*, **21**, 483 (1923).
† Compton, *Phil. Mag.*, **46**, 908 (1923).

argument for two reasons. Firstly, there will be a considerable broadening of the line owing to the Doppler effect, electrons moving away from the incident radiation producing an additional increase in wave-length on scattering, whose magnitude can be shown to be much larger than the Compton effect, while electrons moving towards the incident radiation will produce a corresponding decrease. Secondly, there will be a reduction in the average wave-length of the scattered radiation owing to the fact that the incident radiation is more intense relative to electrons moving towards it, so that these electrons will scatter more than their proper share. The existence of this last effect has been shown by Milne,* who pointed out that if black-body radiation at a certain temperature is incident on free electrons in statistical equilibrium at that temperature, the scattered radiation must, according to the laws of thermodynamic equilibrium, have the same distribution in frequency as the incident radiation.

In the present paper a mathematical investigation is made of these effects. The principle of relativity is used to connect the process of scattering by an electron moving with any velocity to that of scattering by an electron initially at rest. In § 2 an expression is obtained for the spectral distribution of the scattered radiation when radiation of any given spectral distribution is incident on free electrons in statistical equilibrium at a given temperature. In § 3 the special case of an incident line spectrum is considered, the shift and broadening of a line on scattering being obtained. It is found that for the temperature of the sun's atmosphere the half width is about 10 Å., while the resultant shift is much smaller, of the order of the Compton shift 0·024 Å., and may be of either sign, depending on the frequency and the temperature of the electrons. These results are applied to a stellar atmosphere in § 4, it being shown that the effect is the production of wings to the lines, and not an unsymmetrical broadening, as would be required to give a measurable shift. A numerical estimate shows that the intensity of the observed wings of stellar lines cannot be accounted for in this way.

The effect of free electrons on the general opacity of a gas has been considered by Stewart † on the basis of the classical theory. The present paper, based on the light-quantum point of view, does not cover the same ground as Stewart's, being concerned with the effect of the electrons on light of particular wave-lengths near an absorption line, and not with any general effect produced throughout the spectrum.

§ 2. *Deduction of the Spectral Distribution of the Scattered Radiation.*— Consider a single scattering process in which the electron initially has the momentum (m_x, m_y, m_z) while the incident light-quantum has the momentum (g_x, g_y, g_z). Let m_t and g_t be the masses of the electron and light-quantum multiplied by c, the velocity of light. If the direction of the incident quantum is determined by the spherical polar coordinates θ and ϕ, and ν is its frequency, then

$$g_x = h\nu/c \,.\, \cos\theta,\ g_y = h\nu/c \,.\, \sin\theta\cos\phi,\ g_z = h\nu/c \,.\, \sin\theta\sin\phi,\ g_t = h\nu/c \qquad (1)$$

* Milne, *Proc. Phys. Soc. London*, **36**, 100 (1924).

† Stewart, *Phys. Rev.*, **22**, 324 (1923).

The quantities m_x, m_y, m_z, m_t and g_x, g_y, g_z, g_t are the components of two 4-dimensional vectors, the squares of whose "lengths" are given by

$$(m_u, m_u) \equiv m_t^2 - m_x^2 - m_y^2 - m_z^2 = m^2c^2 \quad . \quad . \quad (2)$$

and

$$(g_u, g_u) \equiv g_t^2 - g_x^2 - g_y^2 - g_z^2 = 0 \quad . \quad . \quad . \quad . \quad (3)$$

The suffix u is here used to denote any of the four suffixes x, y, z, and t, and the bracket notation is used to represent scalar products.

The laws of conservation of energy and momentum give

$$m_u + g_u = m_u' + g_u' \quad . \quad . \quad . \quad . \quad (4)$$

where dashed letters are used to denote the corresponding quantities after the scattering process. We can eliminate the quantities m_u' in the following way:

$$(m_u + g_u - g_u',\ m_u + g_u - g_u') = (m_u',\ m_u') = m^2c^2 = (m_u,\ m_u),$$

using (2) and the corresponding relation for dashed letters. Using (3), this reduces to

$$(m_u, g_u) - (m_u, g_u') = (g_u, g_u') \quad . \quad . \quad . \quad (5)$$

This equation determines the frequency ν' of the scattered quantum in terms of that of the incident quantum, the initial momentum of the electron, and the directions of the incident and scattered quanta.

There are only two independent invariants connected with the scattering process, which we may take to be (m_u, g_u) $[=m^2c^2\alpha$ say] and (g_u, g_u') $[=m^2c^2\beta$ say]. Any other invariant that one can form from the four vectors m_u, m_u', g_u, g_u' can be expressed at once in terms of these two by means of the equations (4) and (5).

If we take the incident radiation to be in the direction of the x axis and the scattered radiation to be in the xy plane, equation (5) reduces to

$$m_t - m_x \cos\theta' - m_y \sin\theta' = \nu/\nu' \,.\, (m_t - m_x) - h\nu/c \,.\, (1 - \cos\theta') \quad (6)$$

while α and β have the values

$$m^2c^2\alpha = h\nu/c \,.\, (m_t - m_x), \quad m^2c^2\beta = h^2\nu\nu'/c^2 \,.\, (1 - \cos\theta') \quad . \quad (7)$$

θ' is now simply the angle of scattering.

Suppose now that there is a continuous spectrum of radiation of intensity I_ν per unit frequency range, confined to a small solid angle $d\omega$, incident on dn scattering electrons all with the same momentum (g_x, g_y, g_z). Let $R(\nu')d\nu'd\omega'$ be the number of quanta scattered per unit time into the frequency range ν' to $\nu'+d\nu'$ and into a specified solid angle $d\omega'$. Then we can put

$$R(\nu')d\nu'd\omega' = dn I_\nu d\omega S dg_x' dg_y' dg_z',$$

where S is a certain scattering coefficient whose form is to be determined, and ν is to be regarded as a function of g_x', g_y', g_z' and m_x, m_y, m_z, being the frequency of that incident light-quantum which can be scattered by an $(m_x m_y m_z)$ electron into a $(g_x' g_y' g_z')$ light-quantum. The differentials dg_x', dg_y', dg_z' are determined by $d\nu'$ and $d\omega'$.

$R(\nu')d\nu'd\omega'$ is the frequency of occurrence of a certain event, so it transforms under a Lorentz transformation according to the same law as the frequency of a clock moving with the electrons, which is the same law as m_t^{-1}. Hence $R(\nu')d\nu'd\omega'm_t$ is invariant. Now $I_\nu d\omega$ transforms according to the same law as ν, and the differential $dg_x'dg_y'dg_z'$ according to the same law as g_t' * or ν'. We thus find, since dn is invariant, that $\nu\nu'm_tS$ is invariant, so that it is a function of α and β, as we have seen that these are the only independent invariants connected with the process.

Putting

$$\nu\nu'm_tS=F(\alpha,\beta)/m^2h,$$

which makes F a dimensionless function of the dimensionless quantities α and β, we get

$$Rd\nu'd\omega'=dnI_\nu d\omega F/m^2h\nu\nu'm_t\,.\,dg_x'dg_y'dg_z'.$$

From the relations corresponding to (1) for dashed letters, we find

$$\frac{\partial(g_x'g_y'g_z')}{\partial(\nu'\theta'\phi')}=\left(\frac{h}{c}\right)^3\nu'^2\sin\theta'=\left(\frac{h}{c}\right)^3\nu'^2\frac{d\omega'}{d\theta'd\phi'}.$$

Hence

$$R(\nu')=h^2/m^2c^3\,.\,dnI_\nu d\omega\nu' F/\nu m_t \quad . \quad . \quad . \quad (8)$$

The form of the function F cannot be determined without some special theory of the process of scattering, such as, for example, Compton's assumption that the intensity of the scattered radiation is the same as that given by the classical theory for an electron recoiling with the appropriate velocity,† or Jauncey's corpuscular quantum theory.‡ For visible radiation, these theories give values for the amount of scattered radiation that do not differ appreciably from the classical theory value, which is

$$h\nu R=\tfrac{1}{2}e^4/m^2c^4\,.\,nI_\nu d\omega(1+\cos^2\theta')$$

for n electrons at rest. This gives, with the help of equations (7), in which m_x, m_y, and m_z have been put equal to zero,

$$F(\alpha,\beta)=\frac{e^4}{h^2c^2}\cdot\frac{mc^2}{h\nu}\tfrac{1}{2}(1+\cos^2\theta')=\frac{e^4}{h^2c^2\alpha}\left(1-\frac{\beta}{\alpha^2}+\frac{\beta^2}{2\alpha^4}\right) \quad . \quad (9)$$

approximately. For the present, however, we shall keep F arbitrary.

To generalise equation (8) for the case when there are N scattering electrons in statistical equilibrium at the temperature T, we must replace dn by

$$N(2\pi mkT)^{-\frac{3}{2}}e^{-(m_x^2+m_y^2+m_z^2)/2mkT}dm_xdm_ydm_z,$$

and integrate with respect to m_x, m_y, and m_z. The use of the Newtonian kinetic energy $(m_x^2+m_y^2+m_z^2)/2m$ instead of the relativity one (m_tc-mc^2) in the exponential is an approximation which is quite justifiable even for stellar temperatures.

* Dirac, *Proc. Roy. Soc.*, A, **106**, 583 (1924).
† Compton, *Phys. Rev.*, **21**, 491 (1923).
‡ Jauncey, *Ibid.*, **22**, 233 (1923).

This gives

$$R(\nu')=\frac{h^2}{m^2c^3}N(2\pi mkT)^{-\frac{3}{2}}\int\int\int_{-\infty}^{\infty}\frac{I_\nu d\omega\nu' F}{\nu m_t}e^{-(m_x^2+m_y^2+m_z^2)/2mkT}dm_x dm_y dm_z$$

$$=h^2/m^2c^3\,.\,N(2\pi mkT)^{-\frac{3}{2}}\int_0^{\infty}I_\nu d\omega\psi(\nu,\nu')d\nu \quad . \quad . \quad . \quad . \quad (10)$$

where

$$\psi(\nu,\nu')=\int\int_{-\infty}^{\infty}\frac{\nu' F}{\nu m_t}e^{-(m_x^2+m_y^2+m_z^2)/2mkT}\Big/\frac{\partial\nu}{\partial m_x}\,.\,dm_y dm_z \quad . \quad . \quad . \quad (11)$$

Here $\partial\nu/\partial m_x$ is to be evaluated from (6) and expressed in terms of m_y, m_z, and ν.

Equation (10) gives the contribution of each spectral component of the incident radiation to the scattered radiation of frequency ν'. These contributions are plainly additive, so if the incident radiation consists of a continuous background with an absorption line, the shift and broadening of the scattered line will be just the same as if the incident radiation is simply a bright line of the same frequency.

§ 3. *Broadening and Shift produced by the Scattering.*—Suppose the incident radiation to be monochromatic of intensity $I_\nu d\nu$. The number of quanta of scattered radiation in the frequency interval ν' to $\nu'+d\nu'$ is

$$h^2/m^2c^3\,.\,N(2\pi mkT)^{-\frac{3}{2}}I_\nu d\nu d\omega\psi(\nu,\nu') \quad . \quad . \quad . \quad (12)$$

We may approximate to the value of ψ by eliminating m_x from (11) by means of (6), and expanding the integrand (excluding the exponential part of it) in powers of the dimensionless variables m_y/mc and m_z/mc. On integrating term by term, we shall obtain ψ as a series of ascending powers of the small quantity $(kT/mc^2)^{\frac{1}{2}}$, whose value for T=6000° C. is about 10^{-3}. $h\nu$ may be counted of the same order of magnitude as kT, this being the case when the incident frequency is of the order of magnitude of the frequency of maximum black-body intensity for the temperature T.

Working to the 1st order only, we have from (6)

$$m_x(1-\cos\theta')=-mc(\nu'-\nu)/\nu+m_y\sin\theta'$$

and

$$\partial\nu/\partial m_x=\nu/mc\,.\,(1-\cos\theta').$$

It is easily verified that

$$m_x^2+m_y^2=\frac{2}{1-\cos\theta'}\left[m_y-\frac{mc(\nu'-\nu)\sin\theta'}{2\nu(1-\cos\theta')}\right]^2+\frac{m^2c^2(\nu'-\nu)^2}{2\nu^2(1-\cos\theta')}.$$

Hence, putting

$$a\nu^2=4kT/mc^2\,.\,\nu^2(1-\cos\theta')$$

we get from (11)

$$\psi(\nu,\nu')=\frac{F_0}{\nu(1-\cos\theta')}e^{-(\nu'-\nu)^2/a_\nu^2}\int\int_{-\infty}^{\infty}$$

$$\exp.\left\{-\left[\frac{2}{1-\cos\theta'}\left(m_y-\frac{mc(\nu'-\nu)\sin\theta'}{2\nu(1-\cos\theta')}\right)^2+m_z^2\right]\Big/2mkT\right\}dm_y dm_z$$

$$=\frac{\sqrt{2\pi mkT}F_0}{\nu(1-\cos\theta')^{\frac{1}{2}}}e^{-(\nu'-\nu)^2/a_\nu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where F_0 denotes the value of $F(\alpha, \beta)$ when m_x, m_y, and m_z are put equal to zero.

Expression (12) and equation (13) give the ordinary exponential distribution of intensity for the scattered radiation about the primary frequency, as might be expected from general physical considerations. The intensity falls off to $1/e$ of its maximum value at a distance $\nu'-\nu=a_\nu$, and we may call this the half-width of the line. For T=6000° C., $\theta'=\frac{1}{2}\pi$, a_ν/ν is about 2×10^{-3}. For a wave-length of 5000 Å., this gives a half-width of 10 Å. The greatness of this value is really due to the smallness of the mass of an electron.

The shift $\delta\nu$ of the centre of gravity of the line vanishes to the order of accuracy of equation (13). If it is worked out correct to the second order in $(k\mathrm{T}/mc^2)^{\frac{1}{2}}$, it is found to be, assuming F to have its classical theory value given by (9),

$$\delta\nu/\nu=(4k\mathrm{T}-h\nu)/mc^2\,.\,(1-\cos\theta') \quad . \quad . \quad . \quad (14)$$

The second term in this expression, $-h\nu/mc^2\,.\,(1-\cos\theta')$, represents just the Compton effect for stationary electrons, while the first, $4k\mathrm{T}/mc^2\,.\,(1-\cos\theta')$, is due to a slightly unsymmetrical broadening caused by the motion of the electrons. The two effects are of the same order of magnitude and in opposite directions, so they may just cancel. In any case the shift is of the order of magnitude of the Compton shift 0·024 Å., and since this is of a smaller order of magnitude than the breadth of the line, we should get a different value for the shift if we considered the intensity plotted on a wave-length base than on a frequency base. Such a small shift obscured by such a large broadening could certainly not be detected observationally.

§ 4. *Application to a Stellar Atmosphere.*—The intensity of any spectral component of the radiation from a star is determined by the temperature of the average depth from which radiation of that particular frequency comes. This is given by the mean free path of a light-quantum in the stellar atmosphere, meaning the average magnitude of the vector distance travelled by a quantum without absorption. Any scattering electrons present in the atmosphere will reduce this vector distance. This will cause a general reduction of intensity throughout the spectrum, as worked out by Stewart,* the change in wave-length on scattering not mattering in this case.

There will be an additional effect in the neighbourhoods of absorption lines, owing to the chance of a light-quantum whose wave-length is near an absorption line having its wave-length shifted into the absorption line on scattering, in which case it will be absorbed practically at once. The electrons will thus cause an increase in the effective absorption coefficient of the atmosphere for frequencies near an absorption line, and this will manifest itself by wings attached to the spectral lines. The width of the wings will be given by the maximum change of wave-length on scattering, which has been found to be of the order of 10 Å. No effect of the nature of an unsymmetrical broadening of the lines could be produced, as this would require a large value for the

* Stewart, *loc. cit.*

increase in the absorption coefficient for a short wave-length range just outside the line, falling off rapidly to zero, instead of a fairly uniform value extending for a comparatively large range. The displacement of solar lines to the red near the limb of the sun cannot therefore be accounted for by the Compton effect.

To obtain an estimate of the intensity of the wings produced in this way, we may assume the classical theory value for F given by equation (9). This gives, from (10) and (13), for monochromatic incident radiation of unit intensity,

$$R(\nu')=\frac{Ne^4}{4mc^3h\nu^2(\pi mkT)^{\frac{1}{2}}}\frac{1+\cos^2\theta'}{(1-\cos\theta')^{\frac{1}{2}}}e^{-(\nu'-\nu)^2/a_\nu^2}.$$

If we let N denote the number of electrons per unit volume, $h\nu' R(\nu')$ will be the intensity of the scattered radiation (per unit frequency range) per unit depth of the scattering layer. The chance of a light-quantum being scattered into the frequency range ν_0 to $\nu_0+d\nu_0$ per unit length of its path is now

$$d\nu_0\iint h\nu_0 R(\nu_0)\,.\,\sin\theta' d\theta' d\phi',$$

provided $d\nu_0$ is compared with a_ν. Assuming that the quantum is immediately absorbed when this happens, we find for the length absorption coefficient when ν is very nearly equal to ν_0, the value

$$d\nu_0\frac{22\sqrt{2\pi}}{15}\frac{Ne^4}{mc^3\nu_0(mkT)^{\frac{1}{2}}},$$

corresponding to a mass absorption coefficient

$$k=d\nu_0\frac{22\sqrt{2\pi}}{15}\frac{Ne^4}{\rho mc^3\nu_0(mkT)^{\frac{1}{2}}}$$

where ρ is the density of the stellar atmosphere.

For a temperature of 6000° C. this reduces to

$$k=\frac{170}{M}\frac{Pe}{P}\frac{d\nu_0}{\nu_0}$$

in C.G.S. units, where Pe is the partial electron pressure, P the total pressure of the atmosphere, and M its mean molecular weight. For a solar line $d\nu_0/\nu_0$ is about 10^{-4}, and we may take $Pe/P=\frac{1}{2}$ and M=10 roughly, giving $k=8{\cdot}5\times10^{-4}$.

This is very small compared with the general absorption coefficient in the sun's atmosphere, which, as worked out by Milne,* has the value 1140. The ratio of the reduction in intensity due to the wings to the intensity of the continuous background of the spectrum will be of the order of magnitude of the ratio of these two absorption coefficients, *i.e.* about 10^{-6}. The observed wings of the solar lines can therefore hardly be accounted for by this effect.

§ 5. *Summary.*—A calculation is made of the broadening and shift produced when a spectral line of radiation is scattered by free electrons in statistical equilibrium at a given temperature. It is found that for a temperature of 6000° C. the half-width of the scattered line is about 10 Å., while the shift is very much smaller.

* Milne, *M.N.R.A.S.*, June 1925, p. 770.

The effect of such scattering in a stellar atmosphere is shown to be the production of wings to the lines, and not a shift of the lines. Compton scattering cannot therefore account for the limb effect in the sun.

The intensity of the wings produced in this way is exceedingly small. The observed breadths of solar lines cannot be accounted for by electron scattering.

My best thanks are due to Professor E. A. Milne for much friendly discussion and assistance during the writing of this paper.

Determination of the Solar Parallax from Observations of Mars secured at the Royal Observatory, Cape of Good Hope, near the Opposition of 1924. *I.—Photographic Observations.* By H. Spencer Jones, M.A., D.Sc., and J. Halm, Ph.D.

Determinations of the solar parallax from observations of the planet Mars have been regarded with some suspicion, as peculiarly liable to be affected by errors of a systematic nature. It cannot be denied that suitable minor planets are preferable for the purpose and may be expected to provide a value of the solar parallax with a smaller probable error than will result from observations of Mars ; on a photographic plate the star-like image of a minor planet is a much easier object to measure than the disc-image of a major planet, whilst in heliometer observing, the accuracy with which two star-images can be caused to swing through one another is greater than that with which the image of a star can be set in the centre of the disc of a planet. But apart from the increased accidental error to which the observations of any major planet are liable, there is the danger that the red colour of Mars may give rise to systematic errors, since at moderate zenith distances the effect of chromatic dispersion of the atmosphere cannot be separated from a true parallactic displacement. That the danger is a real one seems to be confirmed by the fact that all the determinations of the solar parallax by meridian observations of Mars have given a larger value than most of the other methods of determining this constant.

Such evidence as is available would appear to indicate that heliometer observations are free from systematic errors arising from atmospheric chromatic dispersion. Gill * was convinced that this was the case, at least in so far as star observations are concerned ; as regards observations of Mars, in the discussion of his observations in 1877, he found on grouping the observations according to altitude no evidence of such an effect.† There is some force in Newcomb's argument : ‡

" It may be objected to the inclusion of Gill's Ascension result that it should be rejected for the same reason, since the colour of the planet would affect heliometer observations and meridian observations

* Gill, *History and Description of the Cape Observatory,* p. lxxvii.
† *Memoirs R.A.S.*, **46**, 157. ‡ *Fundamental Constants of Astronomy,* p. 154.

equally. I have, however, considered it free from the objection in question. . . . In the first place, the result is not too large, but is, on the contrary, the smallest of all the accurate measures. The principle that when a result is open to a strong suspicion of being affected by a cause which would cause it to deviate in one direction, it is logical to conclude *a posteriori* that the cause had not acted if the deviation is found to be in the other direction, may not be a perfectly sound one, but I have nevertheless acted upon it."

Every determination of the solar parallax which can be regarded as free from systematic error is of value in fixing the precise value of this fundamental constant. In view of the unusually close approach of Mars to the Earth at the opposition of 1924, the horizontal parallax at opposition being 23″·60, and of the favourable position of the planet for observation in the southern hemisphere, it was decided to carry out a programme of evening and morning observations visually with the heliometer and photographically with the astrographic refractor for the determination of the solar parallax. It was anticipated that the comparison of the results obtained with the two instruments would prove of interest, particularly with reference to systematic and other errors affecting the observations. The results obtained from the photographic observations are summarised in this paper: the heliometer results are discussed in the following paper.

In order to eliminate as far as possible systematic errors arising from the chromatic dispersion of the atmosphere and the difference of colour of Mars and of the comparison stars, a colour-filter was used to reduce the colour of the planet to approximate equality with that of the stars. At the zenith distance of 60°, at which most of the photographs were secured, a difference of effective wave-length between Mars and the stars of ten tenth-metres will enter systematically into the derived solar parallax by a quantity of the order of 0″·003. Of the screens which were available the most suitable was the Wratten and Wainwright "L" filter. Photographs secured with the laboratory spectrograph showed a fairly narrow band of transmission extending from about λ4250 to λ4530 with a maximum about λ4400. This is approximately the effective wave-length of stars of type G5. Determinations of the effective wave-lengths of stars of various spectral types vary somewhat widely with the instrument employed and the other circumstances of the observations. From the summary and diagram given by Lundmark and Luyten,* it appears that λ4400 is the effective wave-length for type G5 according to Davidson and Martin and also Bergstrand (I.), for type K3 according to Lindblad, and for type F8 according to Wolf. From the whole of the material summarised by Lundmark and Luyten, λ4400 may be accepted as the effective wave-length of stars of type about G5. The absorbing filter used for Mars therefore increased the mean refrangibility of its light into approximate equality with that of the comparison stars, and any residual effects due to chromatic dispersion of the atmosphere should be small and within the probable error of the determination of the solar parallax.

* *M.N.R.A.S.*, **82**, 502, 1922.

The absorbing filter was mounted in front of an adjustable rotating sector which could be rotated at a high speed by compressed air. The opening of the sector was adjusted until it was found by trial exposures that images of the planet of suitable density were obtained with an exposure of 40 seconds, which was the time of exposure adopted throughout. The evening exposures were commenced when the eastern hour-angle of the planet was $4\frac{1}{2}$ hours, or as soon after as conditions permitted. The morning exposures were timed to finish when the planet's western hour-angle was $4\frac{1}{2}$ hours. This hour-angle corresponded to a zenith distance of about 60°. With a few exceptions, the photographs were obtained at hour-angles between $3\frac{1}{2}$ h. and $4\frac{1}{2}$ h. Three exposures were given on each plate and, when conditions permitted, three plates were taken each evening and morning.

For the first few nights, the evening and the following morning exposures were obtained on the same plate, the planet being displaced in right ascension between the successive evening or morning exposures and in declination between the evening and morning. It was found, however, to be more convenient to secure the evening and morning exposures on separate plates, and for the greater part of the programme this procedure was followed. The planet was then displaced between successive exposures in declination. The same scale-readings for setting the planet were used both evening and morning: the displacement between the second and third exposures on each plate was made slightly larger than that between the first and second exposures, so enabling the order of exposure on the plate to be at once recognised.

As it was intended to measure the plates differentially in the Repsold measuring machine, it was necessary that either the evening or the morning plate should be exposed through the glass. In order that the focus setting should remain unaltered throughout the observations, the evening plates were placed in the plate-carrier behind a blank plate from which the film had been cleaned off, whilst the morning plates were exposed through the glass. The earlier plates of the series, which received both evening and morning exposures, were also taken through the glass, in order that the telescope which was in use at the time for securing repetition plates of the Cape Astrographic Zones for the determination of proper-motions could be used for either purpose without alteration of focus. The focus of the instrument, therefore, remained unaltered throughout the whole of the programme.

In order to provide a control over the scale of the telescope, a standard region, with centre at R.A. $22^h\ 16^m\ 38^s{\cdot}3$, Dec. $-14°\ 55'\ 54''$, and having a number of well-distributed bright stars, was selected. One plate of this region was photographed both evening and morning, the plates in all cases being placed film uppermost behind the blank plate used for the evening Mars exposures. A réseau was printed upon these plates after exposure. The standard region was also photographed on one of the evening and on one of the morning Mars plates. The exposure given to this region was in all cases 40 seconds.

The plates were measured in the Repsold measuring machines, a description of which may be found in *M.N.R.A.S.*, **59**, 61–72. Before the measurements were commenced a careful investigation of the

periodic and progressive errors of the micrometer screws of the two machines was carried out. These had previously been investigated in connection with the Carte-du-Ciel work : * the periodic errors were then found to be inappreciable, and the progressive errors, though sensible, could be neglected when the mean readings in opposite orientations of the plate were used. In the present investigation the errors had to be taken into account.

For the determination of the screw errors, a glass scale by Zeiss, marked at intervals of 0·1 mm., was mounted on the focal plane of the measuring microscope, the screws of which have $\frac{1}{2}$ mm. pitch. The screw readings were taken when the spider-thread in the microscope was set successively on the divisions 50·0, 50·5, 51·0, 51·5 . . . 56·0 of the scale, the maximum displacements to be measured on the photographs being not greater than 6 mm. The uniformity of the intervals on the glass scale in the range that was used had previously been tested by comparing the scale in the Zeiss Comparator with the standard engraved silver scale of the latter, the errors of which had been determined at the Physikalisch-Technische Reichsanstalt. Several sets of measurements on the original scale with each screw of the two measuring machines were made. For convenience in subsequent work, two negative photographic reproductions of the scale, on 16 cm. square plates, were made, one of which was used with each measuring machine. The accuracy of setting the spider lines on the bright graduations on a dark field was found to be somewhat superior to the accuracy of setting on the original scale. Several further sets of measures were made using these reproductions ; the relative errors so derived agreed very closely with those derived from the measurements on the scale itself. The mean of all the measurements was used to give the progressive errors which were finally adopted. These errors are given in the following table :—

Progressive Errors of Micrometer Screws.

Screw Reading.	Micrometer II. (J.). Horizontal Screw.	Micrometer II. (J.). Vertical Screw.	Micrometer I. (J.H.). Horizontal Screw.	Micrometer I. (J.H.). Vertical Screw.
	r	r	r	r
1 rev.	0·0000	0·0000	0·0000	0·0000
0 ,,	− ·0043	− ·0045	− ·0036	− ·0029
1 ,,	− ·0052	− ·0051	− ·0067	− ·0050
2 ,,	− ·0077	− ·0066	− ·0102	− ·0072
3 ,,	− ·0093	− ·0099	− ·0133	− ·0090
4 ,,	− ·0098	− ·0108	− ·0141	− ·0095
5 ,,	− ·0099	− ·0109	− ·0136	− ·0098
6 ,,	− ·0101	− ·0119	− ·0131	− ·0106
7 ,,	− ·0083	− ·0102	− ·0129	− ·0109
8 ,,	− ·0058	− ·0087	− ·0107	− ·0089
9 ,,	− ·0030	− ·0048	− ·0085	− ·0062
10 ,,	− ·0025	− ·0035	− ·0052	− ·0032
11 ,,	0·0000	0·0000	0·0000	0·0000

* *Cape Astrographic Zones*, **1**, xiii.–xvii.

It should be noted that as the errors were determined from measurements on scales mounted in the plane occupied by the photographic plate when under measurement, the errors thus determined include both the effects of actual progressive errors in the micrometer screws and of optical distortion of the field of the measuring microscope. No separate investigation of the optical distortion was therefore necessary.

The periodic errors of the screws were investigated in the manner explained on pp. xiii. and xiv. of vol. i. of *Cape Astrographic Zones*, and were found to be negligible in all cases.

The plates were measured differentially, an evening plate and a plate of the following morning being placed film to film, with the morning plate superposed, in the plate-holder of the measuring machine. In connection with the determination of proper-motions of stars in the *C.A. Zones*, the latter had previously been modified so that the orientation of the upper plate relative to that of the lower could easily and rapidly be adjusted. After suitable adjustment, so that the relative displacements of corresponding star images on the two plates were small and the relative orientation of the two plates was also small, the plates were clamped into position. The plates were measured both by Mr. Spencer Jones (J.) and by Dr. Halm (J.H.), using different measuring machines. J. measured with right ascension and J.H. measured with declination as the horizontal co-ordinate. J.'s procedure in measuring was as follows : 5 settings were made with both screws on each of the 6 images of Mars : all the stars with good measurable images were then measured, the differential displacements only being determined ; when these measurements had been completed, 5 further settings were made on the Mars images. J.H.'s procedure was slightly different, in that he made 3 settings on the Mars images at the commencement of the measurements, 3 more when half of the star images had been measured, and another 3 at the completion of the star measurements. The multiplication of the settings on the Mars images reduced the accidental error of setting on a disc, 25″ in diameter at opposition, which is larger than that of setting on a star-disc.

The accurate determination of the scale is very important. The parallactic displacement of the planet between evening and morning is only a fraction of its total displacement, and, therefore, any error in scale value enters into the derived parallax with considerably increased effect. Two separate quantities require to be taken into account : the linear displacement on the plate corresponding to a certain angular displacement in the sky and the value in terms of micrometer-screw revolutions of a linear displacement on the plate. The latter naturally varies with the distance of the photographic film from the microscope objective ; variations in distance which are of the order of the variations in thickness from one plate to another are quite appreciable. The following procedure which, though somewhat indirect, was found in practice to be convenient and satisfactory, was therefore adopted. The photographic plates rest in the plate-holder of the measuring machine on three bearing pins, which are screwed into the plate-holder from beneath. Three brass pieces were constructed carrying counter-

bearing pins, which could quickly be attached to the plate-holder by thumb screws, steady pins fitting into holes in the holder ensuring that the brass pieces took up a definite position, with the counter-bearing pins exactly above the bearing pins in the plate-holder. The counter-bearing pins were adjusted once and for all so that all three were at exactly the same distance from the lower pins, the latter having first been so adjusted that a plate resting on them was accurately in focus at all parts. The two plates to be measured differentially were placed together in the plate-holder, the three brass pieces were superposed, and the bearing pins in the plate-holder were screwed in until the top surface of the upper plate was in contact with the three counter-bearing pins. The top surface of the upper plate was in this way brought into a definite position relative to the microscope objective. After the measurement of the pair was completed, the lower plate was removed and replaced by the photographic reproduction of the Zeiss scale, and the lower bearing pins once more adjusted until the top surface of the upper plate was again brought into contact with the counter-bearing pins. The value of the 5 mm. interval from 50·0 mm. to 55·0 mm. on the scale was then measured on the micrometer screw. In effect, this procedure ensures that the value of a 5 mm. interval on the plate is measured in terms of the micrometer screw under conditions identical to those under which the differential measures of evening and morning plates are secured. In practice, measurements of the value of 5 mm. interval were made with both screws of the micrometer before as well as after the measurement of the two plates.

To the measured differential displacements corrections were applied (i) for the square terms in the differential refraction; (ii) for the difference in right ascension and declination of the centres of the plates.

(i) *Differential Refraction.*—The linear and squared terms in the differential refraction may be represented by

$$\Delta x = ax + by + cx^2 + dxy + ey^2$$
$$\Delta y = \mathrm{A}x + \mathrm{B}y + \mathrm{C}x^2 + \mathrm{D}xy + \mathrm{E}y^2$$

in which the coefficients are given by Kapteyn's formulæ *

$$a = k' \sec^2 q \qquad \mathrm{A} = k' \tan q\,(\tan p + \tan \delta)$$
$$b = k' \tan q\,(\tan p - \tan \delta) \qquad \mathrm{B} = k' \sec^2 p$$
$$c = -a \tan q \,.\, \sin 1' \qquad \mathrm{C} = -a \tan p \,.\, \sin 1'$$
$$d = -(\mathrm{A}+b) \tan q \,.\, \sin 1' \qquad \mathrm{D} = -(\mathrm{A}+b) \tan p \,.\, \sin 1'$$
$$e = -\mathrm{B} \tan q \,.\, \sin 1' \qquad \mathrm{E} = -\mathrm{B} \tan p \sin 1'$$

where k' denotes the constant (photographic) of refraction, and p, q are given by

$$\tan p = \tan \zeta \cos \omega \qquad \tan q = \tan \zeta \sin \omega,$$

ζ being the zenith distance and ω the parallactic angle. The linear terms are included, with differential scale and orientation, in the analysis of the measurements.

* *Bulletin du Comité International Permanent*, t. iii., p. 61, 1902.

The squared terms become :—

in

$$\Delta x \quad -\tan q \,.\, \sin 1'[ax^2+(A+b)xy+By^2]$$
$$\Delta y \quad -\tan p \,.\, \sin 1'[ax^2+(A+b)xy+By^2].$$

Small tables were constructed for deriving p, q, a, b, A, B for the range of declinations and hour-angles used. These corrections require to be added to the measured co-ordinates to free them from the effect of refraction. The differences for the two plates under measurement are applicable to the measured differential co-ordinates.

(ii) *Difference in Plate Centres.*—If two photographs of the same region are obtained under identical conditions except for a displacement of plate centre, it is not possible to superpose exactly corresponding images over the two plates. In the present investigation the scale settings for Mars were the same for evening and morning exposures. The relative displacement of the two plate centres is therefore equal to the total displacement of Mars due to motion and parallax between evening and morning. This amounted to as much as several minutes for some plates, and its effects on the stars away from the centre of the plate are sufficiently large to require to be taken into account.

If $\Delta\alpha$, $\Delta\delta$ denote the differences in right ascension and declination of a star of declination δ and of the plate centre, then *

$$x=\Delta\alpha\cos\delta+\tfrac{1}{2}\cos\delta\sin^2 1'\,.\,\Delta\alpha\Delta\delta^2+A_2'\,.\,\Delta\alpha^3\cos\delta$$
$$y=\Delta\delta+D_1'\Delta\alpha^2+D_2'\Delta\alpha^2\Delta\delta+D_3'\,.\,\Delta\delta^3$$

where

$$A_2'=(1-\tfrac{3}{2}\sin^2\delta)\sin^2 1'$$
$$D_1'=\tfrac{1}{4}\sin 2\delta\,.\,\sin 1'$$
$$D_2'=\tfrac{1}{2}\cos 2\delta\,.\,\sin^2 1'$$
$$D_3'=\tfrac{1}{3}\sin^2 1'.$$

If the co-ordinates of the plate centre are shifted by ξ, η, the orientation and scale remaining unchanged, then $\Delta\alpha$ is changed by $\xi\sec\delta$ and $\Delta\delta$ is changed by η, neglecting squares and products of ξ, η since they are small quantities. It follows from the above expressions for x and y that

$$\Delta x=\xi+\tfrac{1}{2}\xi\,.\,\sin^2 1'\,.\,y^2+\eta\sin^2 1'\,.\,xy+\xi(1-\tfrac{1}{2}\tan^2\delta)\sin^2 1'\,.\,x^2$$
$$\Delta y=\eta+\xi\tan\delta\sin 1'x+\tfrac{1}{2}(1-\tan^2\delta)\sin^2 1'(\eta x^2+2\xi xy)+\eta\sin^2 1'\,.\,y^2$$

where x, y have been substituted for $\Delta\alpha\cos\delta$ and $\Delta\delta$ respectively in the correction terms.

The squared terms, which alone need be considered, may with sufficient accuracy within the range of declination required be written as :—

in Δx

$$(\cdot 95\xi x^2+\eta xy+\cdot 50\xi y^2)\sin^2 1'$$

in Δy

$$(\cdot 45\eta x^2+\cdot 91\xi xy+\eta y^2)\sin^2 1'.$$

These quantities with reversed sign are to be applied to the measured

* *Cape Astrographic Zones,* **1**, xxvi.

differential co-ordinates in order to free them from the effects due to the difference in right ascension and declination of the centres of the two plates which are being compared.

The differential measures of the stars, Δx, Δy, after correction as above are written in the form

$$\Delta x = ax + by + c$$
$$\Delta y = dx + ey + f$$

where x, y are co-ordinates measured from the centre of the plate, a, e include portions due to differences of scale and of differential refraction, b, d include portions due to differences of orientation and of differential refraction.

Aberration terms depend upon the distance of the plate centre from the Sun, and may therefore be assumed the same for evening and morning plates. These equations were solved for a, b, c, d, e, f, the approximate method used in the reduction of the Greenwich astrographic plates being used. It is well known that a rigid least squares solution is very slightly superior in accuracy to the approximate solution and involves a great increase in labour. The quantities a, e give information as to the difference of scale between evening and morning plates. The linear terms due to differential refraction can be extracted from the tables of the refraction coefficients, which were constructed with arguments declination and hour-angle, and removed from the coefficients a and e. The residual quantities may be interpreted as due to differences of scale between evening and morning. When this was done, it was found that the residual quantities were very small and the mean value of the scale differences in the sense evening *minus* morning was found to be $-\cdot00001 \pm \cdot00001$ and, therefore, we may conclude that the scale remains constant from evening to morning.

The scale of the instrument was derived from the measurement of the plates of the standard region previously referred to. Twelve stars, distributed as symmetrically as possible with respect to the plate centre, were selected and their positions were derived from the Washington A.G. Catalogue. These plates were printed with a réseau and were measured and reduced in the same way as the Cape astrographic plates. The provisional scale adopted in the reduction of these plates was

$$1 \text{ mm.} = (1 - \tfrac{1}{300}) \text{ minutes of arc.}$$

The differences between the measured and computed co-ordinates were equated to linear functions of x and y, and after allowing for the linear terms due to differential refraction and aberration, corrections to the assumed scale value were derived. The differential aberration was computed from the formula :—

$$\xi - x = \mathrm{H} \sin 1'' \,.\, x$$
$$\eta - y = \mathrm{H} \sin 1'' \,.\, y$$

where x, y are the measured co-ordinates, ξ, η the co-ordinates corrected for differential aberration, and where also

$$\mathrm{H} = \mathrm{C} (\sin \alpha \cos \delta + \tan \epsilon \sin \delta) - \mathrm{D} \cos \alpha \cos \delta,$$

α, δ being the right ascension and declination of the plate centre, C, D the Besselian star-numbers for the time of exposure, extracted from the *Nautical Almanac*. The mean scale correction derived in this way is $-\cdot00021$, and the mean discordance of each single determination from the mean is $\pm\cdot00005$. Discordances of this magnitude may be attributed to errors of measurement, and constancy of the scale value may therefore be assumed.

The scale value was checked by comparison of photographs secured through the glass of regions in the *Cape Astrographic Zones* with the original photographs of the same region. The new photographs were all secured with the same focus setting that was used for the Mars plates. Thirty-four regions, measurement of which had been completed, were available. From these measurements, made for determination of proper motions, the differential scale of the two plates can be derived, after allowing for the differential refraction effects on the two plates. The scale values for the old plates are given in the volumes of the *Cape Astrographic Zones*. The scale correction obtained in this way is $-\cdot00023$, with a mean discordance of each single determination of $\pm\cdot00007$. As the scale value derived in this way is obtained by differencing two series of measurements, a greater value of the mean discordance than in the direct determination was to be expected. A scale correction of amount $-\cdot00022$ was adopted and regarded as constant throughout the series.

We have, therefore, determined the scale value of the photographs, *i.e.* the value of 1 mm. on the plate in minutes of arc, and also for each pair of plates the value of an interval of 5 mm. on the replica of the Zeiss scale in terms of screw revolutions of the micrometer; the former is a constant quantity, but the latter varies with the thickness of the upper plate and also with slight focal adjustments of the measuring microscope. To complete the reductions it is necessary to determine the true value in millimetres of the 5 mm. interval of the scale. For this purpose the réseau was taken as the basis of comparison. A plate with the réseau printed on it was superposed upon the replica of the scale. The magnification of the measuring microscope was slightly adjusted until a réseau square exactly fitted the reticule in the eye-piece. The interval from one side of the reticule to the other and the value of the 5 mm. interval on the scale replica were determined on the micrometer. Ten pairs of settings were made for both replica and reticule, constituting one set. Twenty sets were made in all, using different réseau squares in order to reduce any slight errors in the réseau.

In this way it was found for the replica used by J. that

$$\begin{aligned} 1 \text{ réseau interval} &= 5 \text{ mm.} = 10^{r}\cdot0051 \pm 0^{r}\cdot0001 \\ 5 \text{ mm. on replica} &= 9^{r}\cdot9821 \pm 0^{r}\cdot0002 ; \end{aligned}$$

5 mm. on the scale replica is therefore actually $5[1-\cdot00230]$ mm. In the same way it was found that for the replica used by J.H., 5 mm. on the scale was actually $5[1-\cdot00232]$ mm. All measurements made on the scales were therefore reduced in these proportions.

The final process of reduction may best be illustrated by an example of a pair of plates, Nos. 1276–1281. Plate 1281 was superposed, and the measures were taken with 1276 as the reference plate. The scale corrections for plate 1276, to the adopted value of 1 mm. $=(1-\frac{1}{3}\frac{1}{0}0)$ minutes of arc, allowing for differential refraction and aberration, were $+\cdot00067$ in x (right ascension) and $+\cdot00024$ in y (declination).

5 mm. on scale replica $=9^{r}\cdot9435$ on horizontal screw
$=9^{r}\cdot9530$,, vertical ,,

But since 5 mm. on scale are actually $5[1-\cdot00230]$ mm.

5 mm. $=9^{r}\cdot9664$ horizontal screw
$=9^{r}\cdot9759$ vertical ,,

The constants of the solution derived from the comparison stars for the three corresponding pairs of exposures are :

$a=$	$+\cdot00027$	$+\cdot00021$	$+\cdot00014$
$b=$	$-\cdot00161$	$-\cdot00161$	$-\cdot00150$
$d=$	$+\cdot00071$	$+\cdot00068$	$+\cdot00074$
$e=$	$+\cdot00013$	$+\cdot00018$	$+\cdot00030$
$c=$	$+0^{r}\cdot5032$	$+0^{r}\cdot4508$	$+0^{r}\cdot4541$
$f=$	$+0^{r}\cdot1214$	$+0^{r}\cdot1607$	$+0^{r}\cdot2252$

The measured Mars displacements, together with the reductions derived from the comparison stars and the total displacements from evening to morning, are :

x.			y.		
Measured.	c.	Total.	Measured.	f.	Total.
r	r	r	r	r	r
9·8838	0·5032	10·3870	1·3025	0·1214	1·4239
9·9433	·4508	10·3941	1·2618	·1607	1·4225
9·9314	·4541	10·3855	1·1919	·2252	1·4171
	Means	10·3889			1·4212

$= 5\cdot2120$ mm. $= 0\cdot7123$ mm.
$-\frac{1}{3}\frac{1}{0}0=-\cdot0174$ $-\frac{1}{3}\frac{1}{0}0=-\cdot0024$
$+\cdot00067=+\cdot0035$ $+\cdot00024=+\cdot0002$
$5'\cdot1981$ $0'\cdot7101$

The total displacement is $5'\cdot2464=314''\cdot78$
,, tabular ,, $=314''\cdot99$

The tabular displacement includes the displacement due to parallax computed on the supposition that the solar parallax is $8''\cdot800$.

If $\Delta\alpha$, $\Delta\delta$ are the tabular displacements in R.A. and Decl. (excluding parallax), p_α, p_δ are the parallax displacements (basis $8''\cdot80$), then the total tabular displacement, D_0, is given by

$$D_0^2=(\Delta\alpha+p_\alpha)^2+(\Delta\delta+p_\delta)^2.$$

If the solar parallax is $8''\cdot80(1+\kappa)$, the true displacement will be given by D, where

$$D^2=(\Delta\alpha+p_\alpha\overline{1+\kappa})^2+(\Delta\delta+p_\delta\overline{1+\kappa})^2=(D_0+\Delta D)^2,$$

whence

$$\Delta D=\frac{1}{D_0}\{p_\alpha(\Delta\alpha+p_\alpha)+p_\delta(\Delta\delta+p_\delta)\}\kappa.$$

The coefficient of κ was computed for each pair of plates measured. For the pair 1276-1281 it is 33·51. Hence from this pair of plates we derive

$$33\cdot51\kappa=-0\cdot21.$$

The coefficient of κ and the tabular displacement were computed for an instant corresponding to the mean time of the three exposures on the plate. This is justifiable, since with three exposures of 40 seconds each one plate could be secured in 5 minutes.

The following table contains the results of the measures. The first column contains the plate numbers, the second contains the number of exposures measured, the third contains the coefficient of κ, $8\cdot80\kappa$ being the correction to the assumed value of the solar parallax. In the fourth column is given the residual, observed *minus* tabular displacement, the mean of the measurements by the two observers being given. The last column contains the relative weights; these are proportional to the number of exposures measured, except that the weight of any combination including a plate which has been used more than once is reduced by one-third. The images on many of the plates are very poor and diffuse, due to the inferior definition prevailing during much of the period covered by the observations. But the weights have not been modified on this account; the residuals appear to be due mainly to guiding error, arising from the considerable difference in magnitude between Mars and the comparison stars; the large residuals are confirmed by both measurers.

Plate No.	Exposures.	Coefficient of κ.	Residual.	Weight.	Plate No.	Exposures.	Coefficient of κ.	Residual.	Weight.
1145	3	20·33	+ ″·08	9	1159	3	25·31	+ ″·02	9
46	3	21·24	− ·41	9	1160	3	24·45	− ·18	9
48	3	8·34	+ ·08	9	61	3	23·60	− ·16	9
49	3	11·09	− ·53	9	68–73	3	27·77	+ ·21	9
1150	3	10·87	− ·43	9	69–72	3	26·95	·00	6
51	3	21·52	− ·27	9	70–72	3	26·50	+ ·02	6
52	3	20·41	− ·07	9	74–79	3	29·28	− ·17	6
54	3	21·82	− ·61	9	76–79	3	27·95	+ ·02	6
55	3	21·08	− ·35	9	75–78	3	28·13	+ ·13	9
56	3	25·14	+ ·07	9	1180–83	3	28·99	+ ·55	6
57	3	23·78	+ ·27	9	80–85	3	29·52	− ·23	6
58	3	21·93	− ·19	9	81–84	3	28·95	+ ·35	9

Plate No.	Exposures.	Coefficient of κ.	Residual.	Weight.
1182–83	3	28·42	+ ″·64	6
82–85	3	28·95	+ ·65	6
86–91	3	31·14	− ·32	9
87–90	3	30·46	+ ·08	9
88–89	3	29·68	·00	9
1192–97	3	31·50	+ ·63	9
93–96	3	30·81	+ ·03	9
94–95	3	29·92	+ ·20	9
98–01	3	31·95	+ ·19	6
99–01	3	31·40	+ ·16	6
1200–01	3	30·86	+ ·12	6
14–18	2	33·68	+ ·44	4
14–19	1	34·81	+ ·22	2
15–18	2	33·30	+ ·36	4
15–19	1	34·43	− ·04	2
16–18	2	32·95	+ ·46	4
16–19	1	34·07	− ·03	2
1227–30	2	35·58	−1·01	4
28–30	2	35·10	− ·35	4
29–30	2	34·63	−1·28	4
1231–35	2	32·57	+ ·44	4
32–35	2	32·22	+ ·79	4
33–35	2	31·84	+ ·70	4
36–41	1	33·96	+ ·55	2
36–43	3	34·60	− ·09	6
37–43	3	33·83	+ ·35	6
37–42	2	33·61	− ·38	4
38–44	3	33·76	− ·35	9
1245–52	3	34·45	− ·24	9
46–51	3	33·61	+ ·14	9
47–50	3	32·51	− ·11	9
1253–60	3	32·43	+ ·17	9
54–59	3	31·78	− ·15	9
55–58	3	31·06	− ·29	9
1269–72	3	32·09	·00	9
68–73	3	32·95	− ·35	9
67–74	3	33·76	+1·04	9
1275–82	3	33·97	− ·43	9
76–81	3	33·51	− ·26	9
1277–80	3	32·98	− ″·48	9
1283–86	2	34·29	− ·88	4
84–86	2	33·92	−1·20	4
88–93	3	32·82	+ ·01	9
89–92	3	32·01	+ ·30	9
87–94	3	33·60	+ ·12	9
1295–02	3	33·74	− ·71	6
96–01	3	33·14	+ ·22	9
97–00	3	32·55	+ ·46	6
97–02	3	33·08	− ·05	6
1305–03	3	27·90	+ ·13	6
06–03	3	28·40	− ·27	6
07–03	3	28·87	+ ·10	6
08–15	3	30·17	+ ·21	9
09–14	3	29·68	·00	9
1310–13	3	29·02	− ·20	9
16–23	3	31·75	+ ·25	9
17–22	3	30·74	− ·12	9
18–21	3	29·61	− ·28	9
1324–30	3	31·09	− ·12	9
24–31	3	31·53	+ ·62	6
25–30	3	30·73	+ ·14	6
26–30	3	30·34	+ ·36	6
26–29	3	29·92	− ·11	6
1340–44	3	25·87	+ ·50	6
39–44	3	26·19	+ ·18	6
39–45	3	26·62	+ ·28	6
1341–44	3	25·50	+ ·38	6
41–43	2	25·00	+ ·45	4
47–54	3	26·02	− ·05	9
48–53	3	25·45	+ ·21	9
49–52	3	24·99	+ ·05	9
1357–53	3	19·42	+ ·51	6
56–53	1	19·74	+ ·70	2
57–59	3	22·16	+ ·46	6
56–59	1	22·52	+ ·41	2
1360–64	3	12·98	+ ·80	6
61–64	3	13·44	+1·03	6
62–64	3	13·90	+ ·74	6

The normal equation for κ is

$$59211\kappa = +59{\cdot}55,$$

whence

$$8{\cdot}80\kappa = +\ 0''{\cdot}009.$$

The probable error for one exposure on one pair of plates (weight 3) is found from the squares of the residuals to be $\pm 0''{\cdot}26$. The probable error of $8{\cdot}80\kappa$ is $\pm 0''{\cdot}005$.

The probable error of the residual derived from one pair of images is large. There are two reasons for this : the accidental error of setting on the disc image of Mars is much larger than that of setting on a small star-image ; several plates give large residuals, exceeding one second, which are confirmed by both measurers, and are apparently of the nature of a guiding error. The residuals have been examined and are found to fit closely a normal error curve. The large number of plates obtained, and the large parallax factors available, combine to reduce the probable error of the derived parallax to a satisfactorily small value.

The resulting value of the solar parallax is

$$8''{\cdot}809 \pm 0''{\cdot}005.$$

This result should be free from systematic errors, except possibly such as might arise from a slight difference in mean refrangibility of the screened light from the planet and of that from the comparison stars. The colour screen used was selected from those available as the one which would bring the mean refrangibilities of planet and stars into closest agreement. The wave-length of the maximum of transmission of the screen corresponds to the effective wave-length of stars which are probably somewhat later in type than the mean of the comparison stars. It may, therefore, be anticipated that any residual effects due to the chromatic dispersion of the atmosphere would tend to increase slightly the derived value of the solar parallax, but the magnitude of any such effect is probably within the probable error of the derived parallax.

Summary.

The solar parallax has been derived from comparison of evening and morning photographs of Mars secured round the opposition of 1924. An absorbing filter was employed to bring the mean refrangibilities of the light of the planet and of the star into approximate agreement. A standard region was photographed at approximately the same altitude for the determination of scale. The resulting value of the solar parallax is

$$8''{\cdot}809 \pm 0''{\cdot}005.$$

Determination of the Solar Parallax from Observations of Mars secured at the Royal Observatory, Cape of Good Hope, near the Opposition of 1924. *II.—Heliometer Observations.* By H. Spencer Jones, M.A., D.Sc., and J. Halm, Ph.D.

In conjunction with the photographic determination of the solar parallax by observations of Mars near its opposition in 1924, a visual determination of this constant was undertaken with the Repsold heliometer of this Observatory.

For this purpose fifteen stars, suitable in position and brightness, were selected in the vicinity of the planet's path, and the planet was compared with each of these stars when within measurable distance on all favourable occasions at times when the planet's hour-angle was between $3^{h}\cdot5$ and $4^{h}\cdot5$ East and West. The comparison between planet and star consisted in setting the star in the centre of the planetary disc by movements of the two objectives in scale and position-angle. Personality in the estimation of the centre of the planet and errors in the adjustments of the objectives were guarded against by the use of a reversing prism mounted in front of the eye-piece, and by observing in opposite positions of the objectives and the position circle.

Distances and position-angles of the arcs joining planet and comparison star were obtained : 108 on 42 evenings and 82 on 31 mornings.

The weather conditions were on the whole unfavourable, especially in the mornings.

An early opportunity was taken of determining the R.A. and Decl. of the comparison stars by meridian observations made with the Reversible Transit Circle of this Observatory. The R.A. and Decl. adopted in the present investigation depend on six observations, three in each position of the instrument East and West.

The derived positions of the stars used are given in Table I. The numbers in the second column are from the Cape Zodiacal Catalogue (1900), except for star *w*, for which is given the number in the A.G. Washington Catalogue. The visual magnitudes are taken from the Draper Catalogue. The star *w* was not originally included, but was observed in one or two instances instead of *o*.

The visual magnitudes range from $5^{m}\cdot7$ to $8^{m}\cdot1$. In order to see the image of the star superposed on the disc of the planet, three absorbing wire-gauze screens of different thicknesses were employed. These were mounted on the segments of a light wheel in front of the objective capable of being rotated from the eye-end of the telescope. The axis of the wheel is divided into six sectors separated by light spokes. Five of these are of equal size, three being covered with the gauze and the other two blank. The remaining sector is twice the size of the others and is also blank. Thus either half of the objective can be covered by any one of the screens, the other being left free, or both can be left free at the same time.

For the determination of the corrections to the instrumental distances and of the index error of the position circle two stars were selected (*f* and *g*) in Table I. The positions of these "standards"

are based on ten meridian observations, the p.e. of each co-ordinate derived from the internal agreement of the observations being $\pm 0''\cdot 10$. The distances and position-angles of the arc between the two standards were observed once during every watch. The observations of these standards supplied sufficient material for examining the dependence of the index-error on the hour-angle in both the eastern and western position of the instrument. Empirical curves showing this dependence were derived, from which the index error for the moment of the planet's observation could be interpolated. The scale-correction, on the other hand, was assumed to have remained constant during the watch.

TABLE I.

Star.	No.	Visual.	Mean R.A. 1924·0.	Mean Dec. 1924·0.
			h m s	° ′ ″
a	3042	6·6	21 52 35·051	−18 15 31·63
b	3043	7·7	52 52·450	19 32 7·05
c	3055	6·4	58 1·003	18 15 8·73
d	3064	7·4	22 3 14·237	19 56 27·50
e	3067	7·8	4 5·088	18 12 21·89
f	3068	7·4	4 17·875	16 54 56·46
g	3070	5·7	4 49·055	18 53 32·58
h	3084	7·1	10 10·527	19 37 31·34
i	3095	8·1	13 24·095	18 32 32·19
k	3099	7·4	15 27·223	17 4 3·98
l	3113	8·1	20 26·639	17 58 43·19
m	3125	6·4	26 13·095	14 58 29·87
n	3136	6·7	32 1·685	16 46 53·86
w	W8443	7·6	40 14·685	15 0 39·60
o	3152	7·2	40 18·745	15 4 31·17
p	3159	5·7	43 40·733	14 27 27·66

Three observers participated in the observations :

J. H. Spencer Jones.
H. J. Halm.
W. A. J. Wilkin.

The same observer observed in the evening and the morning following, so that as far as possible personality should be eliminated in the evening to morning comparisons.

The observed distances and position-angles (Planet—Star) corrected for refraction were compared with the corresponding co-ordinates calculated from the apparent R.A. and Decl. of the comparison star and the R.A. and Decl. of the planet published in the *N.A.* after applying corrections for parallax calculated on the assumption of $\pi = 8''\cdot 800$.

The residuals (O−C) in distance S and position-angle P are generally expressed by equations of the form :

$$\left.\begin{aligned} &S\,.\,s + \sin P \cos\delta(\Delta\alpha - \Delta\alpha') + \cos P(\Delta\delta - \Delta\delta') = O - C \\ &S\,.\,p + \cos P \cos\delta(\Delta\alpha - \Delta\alpha') - \sin P(\Delta\delta - \Delta\delta') = (O-C)\,.\,S \sin 1'' \end{aligned}\right\} \quad (1)$$

Here the terms involving s and p denote corrections on account of the unknown instrumental scale and index error, the quantities $\Delta\alpha$, $\Delta\delta$ the combined effect of the corrections to the tabular right ascension and declination of the planet and those arising from an erroneous assumption of π, and $\Delta\alpha'$, $\Delta\delta'$ corrections to the adopted right ascension and declination of the comparison star.

Similar equations apply to the standards, viz. :

$$\left.\begin{aligned} &S\,.\,s+\sin P\cos\delta(\Delta\alpha_1-\Delta\alpha_2)+\cos P(\Delta\delta_1-\Delta\delta_2)=O-C\\ &S\,.\,p+\cos P\cos\delta(\Delta\alpha_1-\Delta\alpha_2)-\sin P(\Delta\delta_1-\Delta\delta_2)=(O-C)\,.\,S\sin 1''\end{aligned}\right\} \quad (2)$$

$\Delta\alpha_1$, $\Delta\delta_1$, and $\Delta\alpha_2$, $\Delta\delta_2$ being the corrections to the adopted R.A. and Decl. of the Standards.

Eliminating s and p, the four equations (1), (2) lead to equations for $\Delta\alpha$, $\Delta\delta$ of the form :

$$\left.\begin{aligned} \Delta\alpha&=n_\alpha+f_\alpha\\ \Delta\delta&=n_\delta+f_\delta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where n_α, n_δ represent numerical quantities, while f_α, f_δ are expressions depending on the corrections to the adopted R.A. and Decl. of the standards and the comparison star.

If the true solar parallax is represented by

$$\pi=8''\cdot800(1+\kappa),$$

and if p_α, p_δ denote the parallactic effects (apparent—geocentric) in R.A. and Decl., $\Delta\alpha_0$, $\Delta\delta_0$ the corrections to the tabular geocentric co-ordinates, and ϕ_α, ϕ_δ the corrections for phase, we have

$$\left.\begin{aligned} \Delta\alpha+\phi_\alpha&=\Delta\alpha_0+p_\alpha\kappa\\ \Delta\delta+\phi_\delta&=\Delta\delta_0+p_\delta\kappa\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Further, if differences are formed between consecutive evening and morning results based on comparisons of the planet with the same star, we have from (3) and (4),

$$\left.\begin{aligned} (p_\alpha-p_\alpha')\kappa&=(n_\alpha-n_\alpha')+(\phi_\alpha-\phi_\alpha')-(\Delta\alpha_0-\Delta\alpha_0')\\ (p_\delta-p_\delta')\kappa&=(n_\delta-n_\delta')+(\phi_\delta-\phi_\delta')-(\Delta\delta_0-\Delta\delta_0')\end{aligned}\right\} \quad . \quad . \quad (5)$$

where the quantities relating to morning observations are distinguished by the accent.

Provided the corrections to the tabular places of the planet may be assumed to have remained constant during the time-interval, evening-morning, equations (5) can be at once utilised for the determination of κ, since the phase corrections can be calculated from the required data published in the *N.A.*

It may perhaps be admitted that the geocentric positions of the planet are known with sufficient accuracy to assume that the differences $(\Delta\alpha_0-\Delta\alpha_0')$, $(\Delta\delta_0-\Delta\delta_0')$ are negligible for the small time-interval between consecutive evening and morning observations. Unfortunately, however, owing to exceptionally adverse weather conditions, especially in the morning hours, the observations are not so symmetrically distributed that errors of the ephemeris progressing with time may be *a priori* neglected if we wish to utilise the material to its fullest extent.

To obtain an insight into the character of the changes of $\Delta\alpha_0$ and $\Delta\delta_0$ depending on epoch, the following method was adopted. The material was divided into five groups in order of date, and within each group the mean values of $\Delta\alpha$, $\Delta\alpha'$, $\Delta\delta$, $\Delta\delta'$ were formed. Since p_α and p_α' are practically equal and of opposite signs, the arithmetical mean $\frac{1}{2}(\Delta\alpha+\Delta\alpha')$, according to (4), is

$$\tfrac{1}{2}(\Delta\alpha+\Delta\alpha')=\Delta\alpha_0-\phi_\alpha,$$

where the quantities on the right-hand side refer approximately to the moments of meridian passage of the planet.

On the other hand, p_δ and p_δ' being practically equal but of the same sign, we find

$$\tfrac{1}{2}(\Delta\delta+\Delta\delta')=\Delta\delta_0-\phi_\delta+p_\delta\kappa,$$

where $p_\delta\kappa$ is practically constant for all groups.

The values of $\frac{1}{2}(\Delta\alpha+\Delta\alpha')$ and $\frac{1}{2}(\Delta\delta+\Delta\delta')$ are shown in Table II., columns 2 and 3, the phase effects ϕ_α and ϕ_δ in columns 4 and 5, and $\Delta\alpha_0$ and $(\Delta\delta_0+p_\delta\kappa)$ in columns 6 and 7.

TABLE II.

Date.	$\frac{1}{2}(\Delta\alpha+\Delta\alpha')$.	$\frac{1}{2}(\Delta\delta+\Delta\delta')$.	ϕ_α.	ϕ_δ.	$\Delta\alpha_0$.	$(\Delta\delta_0+p_\delta\kappa)$.	No. of Obs.
1924 July 28	+0″·15	−0″·86	−0″·41	−0″·08	−0″·26	−0″·94	28
Aug. 8	−0·01	−1·11	−0·18	−0·01	−0·19	−1·12	26
,, 26	−0·10	−1·03	+0·01	+0·03	−0·09	−1·00	42
Sept. 3	−0·39	−1·22	+0·08	+0·08	−0·31	−1·15	44
,, 12	−0·51	−1·03	+0·24	+0·15	−0·27	−0·88	50

It appears from these results that the corrections to the ephemeris $\Delta\alpha_0$ and $\Delta\delta_0$ are practically constant throughout the whole range of observations.

In order to eliminate the effects of phase and error of ephemeris from the individual values of $\Delta\alpha$ and $\Delta\delta$, smoothed curves representing the quantities $\frac{1}{2}(\Delta\alpha+\Delta\alpha')$ and $\frac{1}{2}(\Delta\delta+\Delta\delta')$ as functions of the date of observation were constructed from the figures of columns 2 and 3. Deducting these from the original $\Delta\alpha$, $\Delta\delta$, fresh residuals n_α, n_δ are obtained which contain only terms depending on the solar parallax and on the corrections to the adopted star places.

Finally, in order to eliminate the latter, the values n_α, n_α' and n_δ, n_δ' were collected in groups according to stars. Mean values of these quantities were formed within each star group. It is easily seen that the mean differences $(\bar{n}_\alpha-\bar{n}_\alpha')$ satisfy the relation

$$(\bar{n}_\alpha-\bar{n}_\alpha')=(\bar{p}_\alpha-\bar{p}_\alpha')\kappa \quad . \quad . \quad . \quad . \quad (6)$$

and that these are the only relations suitable for a determination of κ, since the factors $(p_\delta-p_\delta')$ are practically zero in all cases.

Table III. shows the numerical equations (6) for each star together with their weights (weight unit being assigned to a single observation of $\Delta\alpha$).

Table III.

Star.			Weight.
	$''$	$''$	
a	27κ	$=-0\cdot13$	1·2
b	25	$=-0\cdot67$	0·7
c	30	$=\ 0\cdot00$	4·4
d	25	$=-0\cdot22$	0·5
e	29	$=+0\cdot43$	2·2
f	31	$=-0\cdot46$	3·7
g	32	$=-0\cdot21$	7·7
h	34	$=+0\cdot04$	4·0
i	33	$=-0\cdot37$	1·2
k	34	$=-0\cdot41$	3·7
l	34	$=-0\cdot27$	2·2
m	35	$=-0\cdot05$	1·7
n	33	$=+0\cdot02$	6·0
o, *w*	30	$=-0\cdot34$	3·6
p	29	$=-0\cdot13$	2·5

The solution of these equations leads to the solar parallax:

$$\pi=8''\cdot758\pm0''\cdot011 \text{ (p.e.)}.$$

It will be generally agreed that this value is far too small to be accepted as an irreproachable result. Doubtless some still unknown systematic error must have vitiated the result. At the same time, it is worthy of note that Gill's value derived from heliometer observations of Mars at Ascension in 1877, viz.,

$$\pi=8''\cdot78\pm0''\cdot012 \text{ (p.e.)},$$

is in fair agreement with the above, pointing likewise to a value of π smaller than that which is now considered most probable.

That a small value of the solar parallax should result from both series of heliometer observations of Mars is the more surprising in view of the fact that, *a priori*, it would be anticipated that the effect of the chromatic dispersion of the atmosphere would lead to a large value.

In searching for an explanation of the discrepancy, the first question which arose was whether it might be confined to one of the measured co-ordinates, distance or position-angle only. A separate determination of the parallax from distances only was therefore undertaken. In this case only those instances were selected in which the planet was compared with the same star on consecutive evenings and mornings, it being assumed that the instrumental scale had undergone no change during the interval. The resulting parallax was found to be

$$\pi=8''\cdot762\pm0''\cdot018,$$

almost identical with the value derived from the right ascensions, although naturally less accurate on account of the smaller number of observational data.

The result shows that *an abnormally small value of the solar parallax, viz.* $\pi = 8''{\cdot}76$, *is obtained whether derived from distances, position-angles, or right ascensions.*

Next, an investigation was made whether there were outstanding differences in the values obtained by the three observers.

For this purpose each star group was subdivided into three smaller groups according to observer, and for each of these the mean values of $(\Delta\alpha - \Delta\alpha')$ corrected for phase and tabular error were derived, with the following result:

Observer.	$(\Delta\alpha - \Delta\alpha')$	Weight.	π.
	″		″ ″
J	−0·11	14·7	8·77±0·02
H	−0·38	12·8	8·70±0·02
W	+0·02	14·9	8·81±0·02
Weighted Mean	−0·14	42·4	8·76±0·01

The discrepancies between the three observers considerably exceed the individual probable errors, a result which certainly was not expected considering the comprehensive precautions which had been taken to guard against "personality." To explain the divergences on this hypothesis of personal errors in the setting of the star on the planet, it would have to be assumed that errors with *opposite* signs had been committed in the eastern and western position, an assumption which is clearly an absurdity.

The question then arose whether there was any difference in the method adopted by the observers individually. Now there was indeed one, and only one, point on which the observers appeared to differ, viz. the use of the wire-gauze screens. No attention was given at the time to this point, as it was not realised that the use of screens could have an influence on the observed parallax. A closer consideration, however, now leads to the conclusion that an appreciable effect from this source on the resulting parallax is by no means improbable.

As already mentioned, in order to render the comparison star visible on the planet's disc, the light of the planet had to be reduced by means of a wire-gauze screen. For this purpose the heliometer was provided with three screens of different transparencies. It was left to the judgment of the observer which screen should be used under the given conditions. The observers H and W particularly differed widely in their opinion on this point. While H used the densest screen even in cases where measurements would have been possible with a more open one, W showed great reluctance in using the dense screen. J adopted an intermediate attitude, using the less dense screens for preference, but falling back on the densest screen whenever observations became difficult with the other screens.

Now, owing to atmospheric chromatic dispersion, the disc of the planet formed by the blue portion of its light is nearer the zenith than the disc formed by the red portion. If, therefore, for some reason, one observer receives the impression of a bluer portion of the planet's light than another observer, the zenith distance of the planet, and

consequently the parallax, observed by the former will be less than that observed by the latter.

But it is a well-known fact that the sensibility of the human eye for colour depends on the intensity of the light source. The more the intensity of the planet's light is diminished, the more sensitive will the observer become to the blue portion, and the less sensitive to the red portion of its spectrum (Purkinje's phenomenon). Consequently H, having used in preference the densest screen, should be expected to observe a smaller parallax than W, who preferred the open screen, a conclusion which agrees with the above result. The three observers agree that with the densest screen the ruddy appearance of the planet was completely lost.

The hypothesis here advanced to explain the discrepancies in the results obtained by different observers can fortunately be submitted to other tests.

Thus, in cases of bad definition the densest screen had of necessity to be used more frequently than in instances when the definition was good.

Hence, the zenith distances of the planet derived from observations made under bad conditions of seeing should be found to be smaller than those derived from material obtained under more favourable conditions.

Likewise, faint stars require more frequently the use of the dense screen than bright stars. Hence again zenith distances of the planet derived from faint stars should be on the average smaller than those derived from bright stars.

The material of the right ascensions has been submitted to an exhaustive examination in these different directions, and in each case the relative zenithal displacements of the planet, $\Delta\zeta$, required to account for the observed discrepancies in the right ascensions have been calculated. An exactly similar examination was made in regard to the residuals $\Delta\delta$.

The values of $\Delta\zeta$ derived from the investigation of both right ascensions and declinations are collected in Table IV., which also contains the differences in the values of the solar parallax, $\Delta\pi$, derived under the conditions specified.

Table IV.

	$\Delta\zeta$.		
	From R.A.	From Decl.	$\Delta\pi$.
	″	″	″
J—Mean	+0·02	+0·04	+0·01
H— ,,	−0·13	−0·14	−0·06
W— ,,	+0·10	+0·10	+0·05
Bad—good definition	−0·15	−0·16	−0·08
Faint—bright stars	−0·13	−0·12	−0·06

The agreement between the values of $\Delta\zeta$ derived from right ascensions and declinations is surprisingly close. It demonstrates beyond

doubt that the phenomenon here dealt with has its maximum effect in the direction towards or from the *zenith* and must therefore be attributed to peculiarities in the atmospheric refraction of the planet's light.

The result may be summed up in the statement that *screens used for reducing the light of the planet show a tendency to enhance the susceptibility of the observer for the more refrangible portion of the planet's light and consequently displace the apparent disc towards the zenith as the result of atmospheric chromatic dispersion.*

In consequence of this "physiological" error introduced by the use of screens, the values of π derived under different conditions range within very wide limits. Thus for H observing faint stars under bad definition the resulting parallax is $8''{\cdot}62$, while for W observing bright stars under good conditions of seeing the high value $8''{\cdot}87$ is obtained.

We may safely conclude that if the light of the planet had not been reduced by errors, the derived parallax would have been found to exceed $8''{\cdot}800$ not inconsiderably. This result is not surprising since there is strong reason to assume that the effective wave-length of the undimmed light of the planet is greater than that of the average star, and hence that the corrections for the planet's refraction have been overestimated. By rendering the blue portion of the planet's light more effective the action of the screen is in the opposite direction. Judging by the results, the former effect has been overbalanced by the latter both in Gill's and the present observations, thus leading to an abnormally small value of the solar parallax.

We must admit on these grounds that a reliable determination of the sun's parallax by heliometric observations of Mars is impossible. But although this result is doubtless disappointing, the definite demonstration of the cause of the failure is nevertheless of importance for future research. It has demonstrated the necessity of observing under conditions when the effect of atmospheric chromatic dispersion may be reliably assumed to be the same for planet and comparison stars.

The use of screens must certainly be discarded. The ideal condition to be aimed at should be that the disc of the planet resembles in size the spurious disc of the star, and that the light of planet and star should have the same effective wave-length. The former condition is at least approximately satisfied in the case of minor planets, and hence the fact that the parallax derived from the heliometer observations of Iris, Victoria, and Sappho ($\pi=8''{\cdot}804$) * is in excellent agreement with the photographic results of Eros ($\pi=8''{\cdot}807$) and that derived from the Moon's parallactic inequality ($\pi=8''{\cdot}805$). The conditions of equality of effective wave-length might perhaps be more fully satisfied by the use of a light-filter in front of the eye-piece.

Another fact which this investigation has demonstrated is the high standard of accuracy attained by heliometric measurements. To demonstrate so clearly the existence of such minute discrepancies

* It is significant that the value of the parallax derived from the Iris observations ($8''{\cdot}812$) is larger than those derived from Victoria ($8''{\cdot}801$) or Sappho ($8''{\cdot}798$), and that Iris is the only one of these three planets which is perceptibly yellower than the average star.

as those shown in Table IV. is a performance which reflects the highest credit on the heliometer as a micrometer instrument. Not only has our confidence in the positions of the major planets observed with this instrument been thereby greatly strengthened, but we have also good reasons to expect results of the highest scientific value from the use of the heliometer for the future determination of the solar parallax during the opposition of Eros in 1931.

Summary.

The discussion of the observations of Mars secured with the heliometer at the opposition of 1924 has revealed the existence of a systematic error of physiological origin (Purkinje effect). The differences between the values obtained by the three observers, between those derived from bright and faint stars respectively and between those derived from observations secured under conditions of good and bad definition respectively, can be logically accounted for by this systematic error, which is due to the use of screens for reducing the brightness of the planet.

Though, therefore, the observations under discussion do not furnish a determination of the solar parallax which is of much value, the conclusions derived are of considerable importance for future observations with the heliometer.

Orbital Elements of Mars, Derived from Cape Heliometer Observations, 1899–1924. By H. Spencer Jones, H.M. Astronomer.

Observations of Mars with the heliometer have been secured at the Cape Observatory at every opposition from 1899 onwards. The results are preserved in the form of equations connecting the corrections to the tabular positions of the planets and of the comparison stars, the latter being derived from the "Cape General Catalogue of Stars for 1900·0." The zodiacal stars in this catalogue are now under re-observation with the reversible transit circle; on the completion of this programme of observations, material will be available for the derivation of reliable proper-motions for these stars. A definitive discussion of the heliometer observations can then be undertaken.

The present discussion is, therefore, provisional and supplementary to that made by Hough (*Cape Annals*, vol. viii., pt. vi., p. 4). Hough's discussion was based on the observations secured at the oppositions from 1899 to 1920 inclusive, but owing to an oversight, the change of weight consequent upon the transformation from geocentric to heliocentric co-ordinates was not taken into account for the observations secured at the opposition of 1920. These observations were, therefore, considerably underweighted. Observations have since been secured at the oppositions of 1922 and 1924. A particularly long series of observations were secured at the 1924 opposition, with a view to the derivation

of the solar parallax, and the positions of all the comparison stars used were determined with the reversible transit circle. The place of the planet at this opposition is therefore very well determined.

The corrections to Newcomb's elements of Mars derived from the discussion of the heliometer observations did not agree, except as regards the correction to the eccentricity, with those derived by Ross (*Astronomical Papers of the American Ephemeris*, vol. ix., pt. ii.) from meridian observations. It was, therefore, thought desirable to rediscuss the heliometer results, incorporating the additional material now available, in order to ascertain whether any serious modification to the corrections to Newcomb's elements, previously derived, would result.

Of the 1924 observations, only those secured within ten days of opposition were utilised, so that the approximate formulæ

$$\Delta l,\ \Delta b = \frac{r-\mathrm{R}}{r}(\Delta\lambda,\ \Delta\beta)$$

could be used to derive corrections to the heliocentric longitude and latitude from the corresponding geocentric quantities. r, R denote the radii vectores from the Sun to the planet and the Earth respectively. $(r-\mathrm{R})$ and R are extracted from the *Nautical Almanac*. The positions of the comparison stars for the 1924 opposition derived from observations with the reversible transit circle were reduced to the system of the "Cape General Catalogue of Stars for 1900·0," in order to render the whole of the material homogeneous.

The weights assigned are based upon the geometrical weights of the heliometer figure, the number of sets of observations, and the factor for converting from geocentric to heliocentric corrections.

The results obtained for the different oppositions are summarised in the following table :—

Year.	Δl.	O−C.	$\sqrt{p}$.	Δb.	O−C.	$\sqrt{p}$.	l.
	″	″		″	″		°
1899	+1·16	+0·11	7	−0·09	−0·23	6	118
1901	+0·06	− ·11	11	+ ·12	·00	12	153
03	−0·57	+ ·18	6	+ ·60	+ ·50	6	187
05	−1·58	− ·12	13	+ ·05	− ·03	13	227
07	−1·18	+ ·02	13	− ·02	− ·12	14	284
09	+0·75	− ·07	17	+ ·13	− ·03	17	1
11	+1·74	− ·08	12	+ ·05	− ·13	12	62
14	+1·58	+ ·10	10	+ ·13	− ·02	11	105
16	+0·68	+ ·04	10	+ ·18	+ ·05	12	140
18	−0·35	− ·02	8	+ ·40	+ ·30	7	174
20	−1·16	+ ·07	7	− ·05	− ·13	9	210
22	−1·41	+ ·14	12	·00	− ·09	11	259
24	−0·07	− ·04	34	+ ·18	+ ·05	36	331

In this table, Δl, Δb represent the corrections required to the heliocentric longitude and latitude derived from Newcomb's Tables of Mars. The columns headed (O−C) give the residuals resulting from the

solution below. The quantities $\sqrt{p}$ are the square roots of the relative weights. l denotes the longitude at opposition.

The quantities Δl, Δb were analysed on the assumption that they are expressible in the forms :

$$\Delta l = \mathrm{A} + \mathrm{B}\cos l + \mathrm{C}\sin l + \tfrac{1}{10}\mathrm{DV}$$
$$\Delta b = \mathrm{A}' + \mathrm{B}'\cos l + \mathrm{C}'\sin l.$$

V denotes the perturbation in longitude of Mars due to the action of Venus ; the inclusion of the term containing V is necessary on account of the uncertainty of the mass of Venus.

Analysis by least squares gives the following values of the constants :—

$\mathrm{A} = +0''\cdot104 \pm 0''\cdot019$	$\mathrm{A}' = +0''\cdot130 \pm 0''\cdot024$
$\mathrm{B} = +0\cdot671 \pm \cdot024$	$\mathrm{B}' = +0\cdot033 \pm \cdot030$
$\mathrm{C} = +1\cdot520 \pm \cdot033$	$\mathrm{C}' = +0\cdot035 \pm \cdot038$
$\mathrm{D} = -0\cdot115 \pm \cdot048$	

The residuals from this solution are given in the above table.

It is easily shown that in terms of corrections to the elliptic elements the above coefficients have the values

$\mathrm{A} = \Delta\epsilon$	$\mathrm{B}' = -(\Delta q - p\Delta\epsilon)$
$\mathrm{B} = -2(\Delta k - h\Delta\epsilon)$	$\mathrm{C}' = (\Delta p + q\Delta\epsilon)$
$\mathrm{C} = 2(\Delta h + k\Delta\epsilon)$	

where

$$h = e\cos\Pi, \qquad k = e\sin\Pi$$
$$p = \sin i\cos\Omega, \qquad q = \sin i\sin\Omega$$

ϵ is the longitude of the epoch, Π, Ω the longitudes of perihelion and node respectively, e is the eccentricity, and i the orbital inclination.

The following corrections to the elements result :—

$\Delta\epsilon = +0''\cdot10 \pm 0''\cdot019$	$\Delta i = -0''\cdot002 \pm 0''\cdot033$
$\Delta e = +0\cdot83 \pm 0\cdot016$	$\Delta\Omega = -1\cdot39 \pm 1\cdot05$
$\Delta\Pi = +0\cdot38 \pm 0\cdot11$	

Hough (*loc. cit.*) shows that constant errors $\Delta\alpha_0$, $\Delta\delta_0$ in the system of star places used will give rise to apparent errors in the heliocentric longitudes and latitudes given, with sufficient accuracy, by

$$\Delta l = \cdot310\Delta\alpha_0 + [\cdot138\Delta\delta_0 - \cdot049\Delta\alpha_0]\cos l$$
$$\Delta b = \cdot324\Delta\delta_0 - [\cdot135\Delta\alpha_0 + \cdot050\Delta\delta_0]\cos l$$

Identifying

$$\cdot324\Delta\delta_0 \text{ with } \mathrm{A}'$$

gives

$$\Delta\delta_0 = +0''\cdot40.$$

We then find

$\Delta\epsilon = +0''\cdot104 - 0''\cdot310\Delta\alpha_0$	$\Delta i = -0''\cdot017 - 0''\cdot101\Delta\alpha_0$
$\Delta e = +0\cdot82 + 0\cdot011\Delta\alpha_0$	$\Delta\Omega = -1\cdot80 - 3\cdot07\ \Delta\alpha_0$
$\Delta\Pi = +0\cdot63 - 0\cdot53\ \Delta\alpha_0$	

Δa_0 cannot be separately determined, but can be assumed zero for comparison with Newcomb's elements, as the right ascensions of the stars used are based on Newcomb's equinox determination.

The final corrections to Newcomb's elements are therefore

$$\Delta\epsilon = +0''.104 \pm 0''.019 \qquad \Delta i = -0''.017 \pm 0''.033$$
$$\Delta e = +0.82 \pm 0.016 \qquad \Delta\Omega = -1.80 \pm 1.05$$
$$\Delta\Pi = +0.63 \pm 0.11$$

for a mean epoch 1916.

Also

$$\Delta M/M = \tfrac{1}{10}D = -0.0115 \pm 0.0048.$$

This correction to Newcomb's adopted mass of Venus (1/408,000) is equal in magnitude, but opposite in sign, to that found by Ross. Ross's discussion of the corrections to the elements of Mars was based upon meridian observations from 1751 to 1912. Subdividing into five groups of about thirty years each, three groups gave a positive correction to the mass of Venus, one gave zero correction, and one (covering the modern observations from 1877 to 1912) gave a negative correction amounting to four-tenths of that derived above. Ross's mass of Venus does not fit the heliometer observations; assuming Ross's value and solving the normal equations for A, B, and C, the sum of the squares of the residuals of the resulting solution is 70 per cent. larger than in the solution in which the mass of Venus is derived along with the other elements.

The mass of Venus here derived (1/412,700) is in close agreement with the value used by Leverrier in his tables of Mars (1/412,150). Ross's value, on the other hand, agrees closely with the value derived by Cowell from Greenwich meridian observations (*M.N.R.A.S.*, **66**, 307).

The corrections to the elements of the orbit of Mars may be compared with those derived by Ross, for the epoch 1916 :—

$$\Delta\epsilon = +0''.050 \qquad \Delta i = -0''.146$$
$$\Delta e = +0.86 \qquad \Delta\Omega = +1.62$$
$$\Delta\Pi = -0.99$$

The corrections to the eccentricity are in close agreement, but the corrections to the longitudes of perihelion and node are very divergent. The discordance in the case of the perihelion is increased if Ross's mass of Venus is adopted for the heliometer discussion. The corrections here derived are generally similar to those obtained by Hough from less material, though the differences from Ross are in general somewhat reduced.

The corrections to Newcomb's tabular heliocentric longitude and latitude have been computed from Ross's elements and compared with the heliometer results. The following series of residuals are obtained for the oppositions from 1899 to 1924 respectively :—

In longitude :

+·18 +·09 +·44 +·09 +·02 −·18 +·02 +·39 +·46 +·40 +·33 +·13 −·26.

In latitude :

−·07 +·10 +·52 −·14 −·29 −·06 +·01 +·14 +·18 +·35 −·20 −·25 −·06.

The agreement, particularly in the case of the longitudes, is bad. In the representation of the heliocentric longitude residuals as a harmonic function of the longitude, the coefficient of $\sin l$ derived from the heliometer discussion (1·52) is in tolerably good agreement with that derived from Ross's elements (1·47), but the coefficients of $\cos l$ differ very considerably (0·67 and 0·93 respectively). The discordances cannot be accounted for by suitable choice of $\Delta\alpha_0$ and $\Delta\delta_0$.

In the hope of throwing some light upon these discrepancies the Greenwich meridian observations near opposition from 1901 to 1920 were examined. The accidental errors of the meridian observations are much larger than those of the heliometer observations, but a negative correction to the mass of Venus larger in amount than that derived above was indicated, together with a larger coefficient of $\cos l$ in the analysis of the heliocentric longitude residuals.

Ross (*loc. cit.*, p. 269) draws attention to the "large systematic differences in both co-ordinates between mean results from different observatories for each opposition, which are greater than the probable error of even a *single* determination of the planet's position." In particular, there is a very large personality in determinations of the planet's right ascension. The heliometer observations may be expected to be affected by such errors to a much smaller extent. The discussion of the observations secured at the opposition of 1924 for the determination of the solar parallax shows that the relative personality of the three observers who participated in the observations is of the order of only a few hundredths of a second of arc in both right ascension and declination. It is probable that the discordances between the results derived from the heliometer and meridian observations are due, at least in part, to these systematic errors of personal or instrumental origin, affecting the meridian observations.

The residuals in right ascension and declination of the heliometer observations resulting from the elements derived above are as follows :

	O−C.	
Opposition.	$\Delta\alpha \cos\delta$.	$\Delta\delta$.
	″	″
1899	+0·16	−0·64
1901	− ·26	+ ·10
03	+ ·92	+1·00
05	− ·36	+ ·01
07	+ ·11	− ·43
09	− ·19	− ·20
1911	− ·15	− ·43
14	+ ·25	− ·08
16	+ ·14	+ ·09
18	+ ·25	+ ·71
20	+ ·06	− ·40
22	+ ·47	− ·33
24	− ·21	+ ·08

The residuals are largest for the years 1903, 1918, which are the two years in which the observations have the smallest weight.

Summary.

1. From a discussion of Cape heliometer observations of Mars at oppositions from 1899 to 1924, corrections to Newcomb's elements (epoch 1916) are derived as follows :—

$$\Delta\epsilon = +0''\cdot104 \pm 0''\cdot019 \qquad \Delta i = -0''\cdot017 \pm 0''\cdot033$$
$$\Delta e = +0\cdot82 \pm 0\cdot016 \qquad \Delta\Omega = -1\cdot80 \pm 1\cdot05.$$
$$\Delta\Pi = +0\cdot63 \pm 0\cdot11$$

2. A correction is determined to the mass of Venus, given by

$$\Delta M/M = -0\cdot0115 \pm \cdot0048.$$

3. With the exception of the eccentricity, these values differ from the corrections derived by Ross by quantities larger than would be anticipated from the assigned probable errors. In particular, the corrections to the mass of Venus, though equal in amount, are opposite in sign.

4. The explanation of these discordances is probably to be found in the systematic errors to which the meridian observations of Mars are liable.

On Horrocks's Treatment of the Evection and the Equation of the Centre, with a Note on the Elliptic Hypothesis of Albert Curtz and its Correction by Boulliau and Newton. By S. B. Gaythorpe.

The importance, historically, of the lunar hypothesis of Jeremiah Horrocks (1619 ?–1641), and his explanation of the evection "by means of a libratory motion of the apsides and a variable eccentricity," are well known from the words of Professor Grant (*Hist. of Phys. Astron.*, London [1852], p. 424); but how far that hypothesis corresponds with modern theory is not immediately apparent.

The terms in the moon's longitude which represent, to the second order of approximation, the elliptic inequality and the evection, may be written :—

$$\mathfrak{C} = L + 2e \sin(L-\Pi) + \tfrac{5}{4}e^2 \sin 2(L-\Pi) + \epsilon \sin(L-\Pi+2(\Pi-L')) \quad (1)$$

where e is the moon's eccentricity; ϵ, the coefficient of the evection; and L, L′, and Π are the mean longitudes, at the given time, of the moon, the sun, and the lunar perigee, respectively.

If we assume

$$\mathfrak{C}\cos\delta = e + \tfrac{1}{2}\epsilon \cos 2(\Pi-L') \quad . \quad . \quad . \quad (2)$$

and

$$\mathfrak{C}\sin\delta = \tfrac{1}{2}\epsilon \sin 2(\Pi-L') \quad . \quad . \quad . \quad . \quad (3)$$

then, as is well known, the second and fourth terms of equation (1) may be combined into a new term, $2e \sin (L-\Pi+\delta)$; while to the same order of approximation the third term may be written

$$\tfrac{5}{4}e^2 \sin 2(L-\Pi+\delta).$$

Hence,

$$\mathfrak{C}-L=2e \sin (L-\Pi+\delta)+\tfrac{5}{4}e^2 \sin 2(L-\Pi+\delta) \quad . \quad . \quad (4)$$

which, obviously, is the equation of the centre in a variable ellipse, of which the eccentricity at the given moment is e, and the longitude of the nearer apse, $\Pi-\delta$.

It is easy to see the geometrical significance of equations (2) and (3): namely, that the centre of the moon's orbit is moving in an epicycle of radius $\frac{1}{2}\epsilon$ with double the angular velocity with which the sun separates from the apse, the centre of the epicycle being situated on the mean line of apsides at a distance from the lower focus equal to the mean eccentricity.

Dividing (2) by (3) gives :—

$$\cot \delta=\frac{2e}{\epsilon} \operatorname{cosec} 2(\Pi-L')+\cot 2(\Pi-L') \quad . \quad . \quad (5)$$

By differentiating this equation, or, more simply, by considering the above geometrical construction, we obtain as the maximum value of δ, $\Delta=\sin^{-1}\frac{\epsilon}{2e}$, and as the condition when that value is reached, $\cos 2(\Pi-L')=-\frac{\epsilon}{2e}$.

After simplifying, equation (5) becomes :—

$$\begin{aligned}\cot \delta&=\frac{2e+\epsilon}{2\epsilon}\cot (\Pi-L')+\frac{2e-\epsilon}{2\epsilon}\tan (\Pi-L'),\\ &=\frac{1+\sin\Delta}{2\sin\Delta}\cot (\Pi-L')+\frac{1-\sin\Delta}{2\sin\Delta}\tan (\Pi-L'),\\ &=\frac{1+\dfrac{1-\sin\Delta}{1+\sin\Delta}\tan^2(\Pi-L')}{\left(1-\dfrac{1-\sin\Delta}{1+\sin\Delta}\right)\tan(\Pi-L')}.\end{aligned}$$

But

$$\frac{1-\sin\Delta}{1+\sin\Delta}=\cot^2(45^\circ+\tfrac{1}{2}\Delta),$$

whence

$$\begin{aligned}\tan\delta&=\frac{\tan(\Pi-L')-\cot^2(45^\circ+\frac{1}{2}\Delta)\tan(\Pi-L')}{1+\cot^2(45^\circ+\frac{1}{2}\Delta)\tan^2(\Pi-L')},\\ &=\tan[\Pi-L'-\tan^{-1}(\cot^2(45^\circ+\tfrac{1}{2}\Delta)\tan(\Pi-L'))],\end{aligned}$$

or

$$\delta=\Pi-L'-\tan^{-1}(\cot^2(45^\circ+\tfrac{1}{2}\Delta)\tan(\Pi-L')) \quad . \quad . \quad (6)$$

which is the equivalent of Horrocks's precept for finding the equation of the apse (*Opera Posthuma*, Londini, 1673, p. 469, § 1).

The eccentricity of the instantaneous ellipse is now found from equation (2). In his corresponding precept, however, Horrocks (*loc. cit.*, § 3) omits the factor cos δ, which of course never differs much from unity. But this omission, and possibly also his disregard of the geometrical "unities" by using one epicycle to find δ and another of slightly larger radius for ϵ, were apparently to cost him dear, for Newton (*Principia*, ed. tert., 1726, pp. 461–2), while acknowledging Horrocks as the first to determine the moon's orbit to be an ellipse, gives Halley the credit of locating the centre on an epicycle.*

The eccentricity, ϵ, and the position of the apse, $\Pi-\delta$, at the given time having thus been found, the equation of the centre must next be determined. The method which Horrocks adopted (*Op. Posth.*, p. 469, §§ 4 and 5), and seems to have invented, depends on the fact that, while Kepler's equation, which may be written $E-M=e \sin E$, does not, when E is unknown, admit of a direct trigonometrical solution, another equation of similar form, $\sin (E'-M)=e \sin E'$, is easily solved by :—

$$\tan (E'-\tfrac{1}{2}M)=\frac{1+e}{1-e} \tan \tfrac{1}{2}M \quad . \quad . \quad . \quad (7)$$

If for e we substitute $\sin \phi$, the right-hand member becomes $\tan^2 (45^\circ+\frac{1}{2}\phi) \tan \frac{1}{2}M$, which is the form that most nearly corresponds to that used by Horrocks, who, however, in accordance with the custom of that time, reckoned the anomaly from the further apse.†

It is obvious that, when e is small, E′ will differ from E by only a very small quantity, and will thus afford, as is now well known, a very useful first approximation to the eccentric anomaly. Horrocks, indeed, regards this result as being of itself sufficiently exact, and proceeds to calculate from it the true anomaly in the usual way, using ϵ, of course, throughout as the eccentricity. ‡

The error of this method of determining the equation of the centre may be investigated as follows. If $\tan y=n \tan x$, then by a well-known method of expansion, $y=x+\frac{n-1}{n+1} \sin 2x+\frac{1}{2}\left(\frac{n-1}{n+1}\right)^2 \sin 4x+ \ . \ . \ .$, whence from equation (7), to the cube of e,

$$E'=M+e \sin M+\tfrac{1}{2}e^2 \sin 2M+\tfrac{1}{3}e^3 \sin 3M.$$

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* It is perhaps worth noting that the difference between the radii of the two epicycles used by Horrocks, 0·01162−0·01129=0·00033, is very nearly equal to the radius of the small additional epicycle, 0·000352, introduced by Newton (*op. cit.*, p. 463).

† For the sake of comparison, the methods and formulæ of the older astronomers referred to in this paper have been modified, where necessary, so as to bring them into conformity with modern usage in this respect.

‡ Horrocks also uses ϵ and E′ to determine the moon's radius vector, and was thus able to represent with considerable accuracy the combined effect on that co-ordinate of the elliptic inequality and the evection.

As in the longitude further corrections are introduced for the annual equation and the variation, and as the latitude is found by Tycho Brahe's hypothesis of a variable inclination and a libratory motion of the nodes, the moon's place according to Horrocks may be said to include the equivalent of all terms of the second order, except that in the radius vector arising from the variation.

Now let $\tan \frac{1}{2}v' = \sqrt{\frac{1+e}{1-e}} \tan \frac{1}{2}E'$, so that v' is the true anomaly corresponding to an eccentric anomaly E'; and let $n = \sqrt{\frac{1+e}{1-e}}$, then $\frac{n-1}{n+1} = \frac{e+\frac{1}{8}e^3+\ldots}{2-\frac{1}{4}e^2-\ldots} = \frac{e}{2}+\frac{e^3}{8}+\ldots$; whence to the same order of approximation,

$$\tfrac{1}{2}(v'-E') = \left(\frac{e}{2}+\frac{e^3}{8}\right)\sin(M+e\sin M+\tfrac{1}{2}e^2\sin 2M)+\frac{1}{2}\left(\frac{e}{2}\right)^2\sin 2(M+e\sin M)+\frac{1}{3}\left(\frac{e}{2}\right)^3\sin 3M,$$

and after simplifying,

$$v'-E' = e\sin M+\tfrac{3}{4}e^2\sin 2M+2e^3\sin M-\tfrac{17}{6}e^3\sin^3 M.$$

But by hypothesis

$$E' = M+e\sin M+\tfrac{1}{2}e^2\sin 2M+e^3\sin M-\tfrac{4}{3}e^3\sin^3 M,$$

whence by addition

$$v' = M+2e\sin M+\tfrac{5}{4}e^2\sin 2M+3e^3\sin M-\tfrac{25}{6}e^3\sin^3 M.$$

But the true anomaly, v, corresponding to M, is

$$v = M+2e\sin M+\tfrac{5}{4}e^2\sin 2M+3e^3\sin M-\tfrac{13}{3}e^3\sin^3 M,$$

whence

$$v'-v = \tfrac{1}{6}e^3\sin^3 M \quad . \quad . \quad . \quad . \quad (8)$$

which, to the order of approximation above mentioned, is the error of Horrocks's method of determining the equation of the centre.

With this it may be interesting to compare an alternative method of finding approximately the equation of the centre, which for nearly a century after Horrocks's death (1641) was a favourite with astronomers, and the further improvement of which engaged the attention even of Newton.

The simple elliptic hypothesis, though frequently ascribed to Seth Ward (1617–1689) from his having adopted it in his *Astronomia Geometrica*, Londini, 1656, seems to have been first proposed by the Jesuit Albert Curtz (*Latine*, Curtius, 1600–1671), a professor of mathematics and philosophy in different houses of his order, and fellow, and afterwards rector, of the Jesuit College at Dillingen, where in 1626 was published his *Novum Cœli Systema*, in which presumably his elliptic hypothesis first appeared. But the book apparently being extraordinarily scarce—even Dr. Frisch, when preparing his edition of Kepler, had not seen a copy (*Kepl. Op. Omn.*, vi. (1866), 583),—the chief sources of information are Curtz's letter to Kepler of 1627 July 19 (*loc. cit.*), and the latter's graceful reference in *Tabulæ Rudolphinæ* (Ulmæ, 1627, p. 77) to the hypothesis of his "Uranian friend." From these it appears that Curtz accepted Kepler's first law, but in place of the second

assumed that the body moving in the ellipse describes equal angles in equal times about the upper, or "empty" focus, whence we arrive at the following construction.

About the "empty" focus H describe an angle AHP = the mean anomaly, M, then the elliptical arc AP will subtend at the lower focus, S, an angle ASP = V, the "true" anomaly according to the hypothesis. Produce HP to K, making PK = PS. Join KS. Then in the triangle HKS, HK = $2a$, HS = $2ae$, and by the tangent formula in trigonometry, $\tan(\mathrm{ASK}-\frac{1}{2}\mathrm{M})=\frac{1+e}{1-e}\tan\frac{1}{2}\mathrm{M}$, whence, to the cube of e, by the same expansion as before,

$$\mathrm{ASK}-\mathrm{SHK}=e\sin\mathrm{M}+\tfrac{1}{2}e^2\sin 2\mathrm{M}+\tfrac{1}{3}e^3\sin 3\mathrm{M}.$$

But, by geometry,

$$\mathrm{ASK}-\mathrm{SHK}=\mathrm{HKS}=\tfrac{1}{2}\mathrm{HPS}=\tfrac{1}{2}(\mathrm{V}-\mathrm{M}).$$

Hence,

$$\mathrm{V}-\mathrm{M}=2e\sin\mathrm{M}+e^2\sin 2\mathrm{M}+2e^3\sin\mathrm{M}-\tfrac{8}{3}e^3\sin^3\mathrm{M} \quad . \quad (9)$$

which is the equation of the centre according to the hypothesis of Curtz.*

But the true equation of the centre to the same order of approximation is

$$v-\mathrm{M}=2e\sin\mathrm{M}+\tfrac{5}{4}e^2\sin 2\mathrm{M}+3e^3\sin\mathrm{M}-\tfrac{13}{3}e^3\sin^3\mathrm{M},$$

whence by subtraction,

$$\mathrm{V}-v=-\tfrac{1}{4}e^2\sin 2\mathrm{M}-e^3\sin\mathrm{M}+\tfrac{5}{3}e^3\sin^3\mathrm{M} \quad . \quad . \quad (10)$$

which, to the cube of e, is the error of the elliptic hypothesis of Curtz.

In order to obtain a closer approximation, Ismael Boulliau (*Latine*, Bullialdus, 1605–1694), in his *Astronomiæ Philolaicæ Fundamenta*, Parisiis, 1657, proposed to take as the planet's true place a point P′, whose position is defined by the intersection of the ellipse with a straight line joining H and the point in which the auxiliary circle is cut by the ordinate passing through the point P. Boulliau's method of calculating the angle ASP′ is long and involved, but Nicholas Greenwood, of whom nothing seems to be known except his *Astronomia Anglicana*, London, 1689, reached the same result by taking as the mean anomaly, instead of M, an angle M′ found from the proportion $\tan\mathrm{M}' : \tan\mathrm{M} :: a : b$, so that $\tan\mathrm{M}'=\dfrac{\tan\mathrm{M}}{\sqrt{(1-e^2)}}$.

If we put

$$n=\frac{1}{\sqrt{(1-e^2)}}, \quad \text{then } \frac{n-1}{n+1}=\tfrac{1}{4}e^2+\tfrac{1}{8}e^4+\ldots,$$

---

* Except for the use of the tangent formula, the above geometrical construction is due to Curtz, who appears, from the letter to Kepler before mentioned, to have noted the close correspondence, not only of the angle ASP and the true anomaly, but also of the angle ASK and the eccentric anomaly. The latter approximation is not referred to by Kepler in his note on Curtz, and the use which Horrocks made of it seems to have been independent and original.

and thus by the same expansion as before, neglecting powers of e beyond the cube, we get $M'=M+\frac{1}{4}e^2 \sin 2M$. Substituting for M this value of M′ in the series (9) for V−M, we have to the same order of approximation,

$$V'-M-\tfrac{1}{4}e^2 \sin 2M=2e \sin (M+\tfrac{1}{4}e^2 \sin 2M) \\ +e^2 \sin 2M+2e^3 \sin M-\tfrac{8}{3}e^3 \sin^3 M,$$

which, after simplifying, becomes

$$V'-M=2e \sin M+\tfrac{5}{4}e^2 \sin 2M+3e^3 \sin M-\tfrac{11}{3}e^3 \sin^3 M.$$

But

$$v-M=2e \sin M+\tfrac{5}{4}e^2 \sin 2M+3e^3 \sin M-\tfrac{13}{3}e^3 \sin^3 M,$$

whence

$$V'-v=\tfrac{2}{3}e^3 \sin^3 M \quad . \quad . \quad . \quad . \quad (11)$$

which, to the cube of e, is the outstanding error of Boulliau's correction of the elliptic hypothesis. It will be seen that this error, though of the same order as that of Horrocks's method (8), is numerically four times greater.

Lastly, in the *Principia* (ed. sec., 1713, pp. 103–4; ed. tert., pp. 111–2), Newton shows how the approximation afforded by the elliptic hypothesis can be made still more exact by the application of two corrections to the mean anomaly instead of one. The maximum value of the first correction is defined as $\sin^{-1}\frac{(a+b)(b-p)}{(2a)^2}$, or $\sin^{-1}(\frac{1}{4}e^2-\frac{1}{8}e^4-\ldots)$, and that of the second as $\sin^{-1}\frac{4ae(b-p)}{3a^2}$, or $\sin^{-1}(\frac{2}{3}e^3-\frac{1}{6}e^5-\ldots)$. In order to find the particular corrections to be added to the mean anomaly in each case, the above maximum values are to be multiplied by $\sin 2M$ and $-\sin^3 M$, respectively. It will at once be seen that the second correction serves to remove the outstanding error of Boulliau's hypothesis. But the approximation is even more exact, for Newton states that the error will scarcely exceed 1″ of arc even in the case of Mars, which seems to imply that the method is correct at least to terms of the fourth order, as, for that planet, the numerical coefficient of $\sin 5M$ in the series for $V-v$ is only about 1″·07.

It is interesting to compare with the above the statement in the first edition of the *Principia* (1687, pp. 113–4). The maximum values of the two corrections are there given as $\tan^{-1}\frac{a-b}{a+b}$, or $\tan^{-1}(\frac{1}{4}e^2+\frac{1}{8}e^4+\ldots)$, and $\tan^{-1}\frac{2ae(a-b)}{3b(a-\frac{1}{4}p)}$, or $\tan^{-1}(\frac{1}{9}e^3+\frac{5}{27}e^5+\ldots)$; and these are to be multiplied by $\sin 2M$, and $-\text{haversine } 2M$, or $-\sin^2 M$, respectively. The first of these corrections is obviously equivalent to Boulliau's, but the significance of the second is not immediately apparent. As the outstanding error of Boulliau's approximation, $\frac{2}{3}e^3 \sin^3 M$, is by hypothesis a very small quantity, it may perhaps be allowable to substitute for it $\frac{2}{3}e^3 \sin^2 M \times$ mean value of $\sin M$. But between the limits $M=0°$ and $M=180°$ the latter factor is $\frac{2}{\pi}$, or approximately $\frac{2}{3}$. Hence the

second correction to M becomes $-\frac{1}{9}e^3$ haversine 2M, in accordance with Newton's precept.*

As, apart from Newton's improvement, Boulliau's hypothesis was for many years highly esteemed as a method of calculation, and the planetary tables of Streete (*Astronomia Carolina*, London, 1661), founded upon it, were long regarded as the best extant, it is perhaps not so surprising that Horrocks's approximation should apparently have failed to attract contemporary attention, or even to become known by his name. But after Newton's method of continuous approximation had appeared in the *Principia* (lib. i., prop. xxxi., schol.), it was obviously only a question of time before computers would endeavour to shorten that process by the use of some simple and fairly close first approximation to the eccentric anomaly, such as Horrocks's, instead of beginning more or less vaguely—"per constructionem quamvis, vel utcunque conjecturam faciendo." The earliest reference bearing on the matter that I have been able to trace in Delambre—or elsewhere—is in his article (*Hist. de l'Ast. au 18^e siècle*, Paris, 1827, p. 186) on Le Monnier, who in a note in *Institutions Astronomiques* (Paris, 1746, p. 500 *n.*; *cf.* also p. 479 *n.*) credits [Jakob] Herman[n, 1678–1733, pupil of James Bernoulli, and author of *Phoronomia*, Amstelædami, 1716] with having abridged the method of Newton by taking the angle E′ given by equation (7), or rather a corresponding expression, as the starting point for the approximation.† Delambre adds: "Cette abréviation commode a été généralement adoptée."

Nevertheless the method seems once more to have been lost sight of; but in comparatively recent years attention has again been directed to its possibilities by Professor H. A. Howe and others. In December 1919 further developments affording solutions of a very high order of approximation were contributed by Professor H. C. Plummer to *M.N.*, **80**, 207, and it may perhaps be permissible to add, about the same time, but independently, by myself to *B.A.A.J.*, **30**, 94.‡ As it was not then known that the idea of the approximation was first due to Horrocks, it was unfortunately not possible to do him the tardy justice of an attribution.

During the past few years, however, through the kindness of the Astronomer Royal, I have devoted a good deal of time, in connection with my work on Horrocks, Crabtree, and Gascoigne, to investigating the MS. matter relating to those astronomers in the Record Room of the Royal Observatory; and in a holograph draft of Horrocks's, entitled "Philoſophicall Exerciſes," I have come across the following

* The first edition of the *Principia* also states that from the mean anomaly thus corrected the true anomaly may be found "per methodum notissimam Dris. *Sethi Wardi* Episcopi *Salisburiensis* mihi plurimum colendi." In the second edition (p. 104) the sentence ends with "notissimam," but "*Wardi*" is inserted before "methodum." In the third edition (p. 112) the sentence likewise ends with "notissimam," but "*Wardi*" is omitted.

† Hermann's paper, which is dated 1726 Feb. 26, is printed in *Commentarii Academiæ Scientiarum Imperialis Petropolitanæ*, Tom. I (ad. ann. 1726), Petropoli, 1728, pp. 142–8. There is, however, no mention of Horrocks, and it is obvious that Hermann's work was quite independent.

‡ Three misprints in the latter paper were corrected in the next issue, *B.A.A.J.* (1920 January), **30**, 150.

passages in Part 2, § 3, which I here venture to quote, as they appear strongly to support the attribution to him of the approximation.

"from this greateſt phyſicall æquation [*i.e.* the arc whose circular measure is e], Kepler deduceth the particular æquation for every degree of anomalia eccentri, ſo finding thence anomalia media; but for to worke converſly he hath no direct way, neyther do J thinke any can be given: yet if inſtead of y^{e} number of ſeconds, he would vſe y^{e} ſine of y^{e} greateſt equation, and frõ thence find y^{e} ſine of each particular equation (which will meerly inſenſibly differ from his method) we may from anomalia media find anomalia eccentri viâ directâ. . . ."

Then follows a geometrical demonstration, ending with a precept for calculation, equivalent to equation (7), and corresponding to the Latin precept in *Op. Posth.* (p. 469, § 5), after which he proceeds :—

"The difference which is betwixt Keplers calculation and this is merely inobſervable, in ♂ it is but 10″ at moſt, in ☉ & ♀ not 1″, in ♄ & ♃ but 3″ or 4″, in ☿ it is 1′ 40″ ſometimes, w^{ch} yet in his apparent place will not cauſe above 50″ difference w^{ch} is iuſt nothing. . . . The cauſe of y^{e} difference is y^{t} Kepler reckons y^{e} length of y^{e} arch of y^{e} phyſicall equation, J the ſine, which increaſe not proportionably, & yet in ſmall arches with inſenſible difference. . . ."

Horrocks's estimate of the error is for the most part too small. Its approximate value from equation (8) is for ♂ 28″, ☉ 0″·2, ♄ 6″, and ☿ 5′.

In conclusion, I should like to express my appreciation of the helpful suggestions I have received from the Astronomer Royal and Dr. Jackson.

The Motions and the Distances of Spiral Nebulæ.

By Knut Lundmark, Ph.D. (Plates 18, 19.)

At present there are several different ways in which the distances of the spiral nebulæ can be estimated. The different methods to be used are properly divided into a direct and an indirect group. The former includes the rather few methods where no assumption is made concerning the physical nature of the spiral nebulæ. The latter embraces the many different methods where some assumption is made, for instance, that the spirals are agglomerations of stars, replicas of our stellar system, masses of gases or dust, or that individual objects observed in the spirals, such as novæ, cepheids, ordinary stars, and gaseous or dark nebulæ have, on the average, the same properties as the corresponding objects in the stellar system.

A. Direct Methods.

(a) *Measures of Parallaxes.*—I think it can justly be said that the attempts to measure the parallaxes of spiral nebulæ have shown that

these objects are out of reach for the present methods of direct parallax-measures.*

(b) *Proper Motions and Radial Velocities.*—The use of these data indicates a way of obtaining stellar distances which is fairly free from objections. The main difficulty in this method is the estimation of the influence of the different sources of error. In the case of spiral nebulæ this influence is very important, as the proper motions seem to be affected by errors of quite the same order of magnitude as the motions themselves.

Mainly on account of the enthusiastic and skilful work of V. M. Slipher we have now knowledge of 44 radial velocities of spiral nebulæ

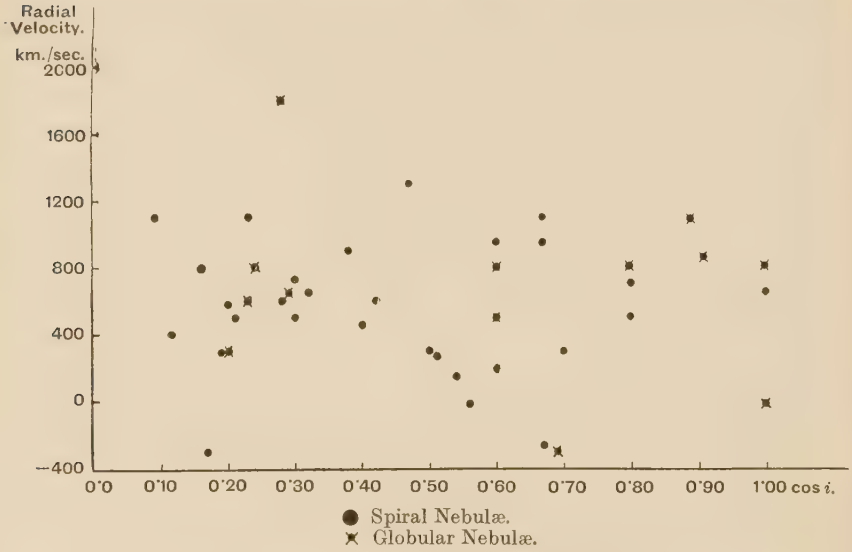

FIG. 1.—Relation between Inclination of Spiral Nebulæ (i) and Radial Velocity.

or related objects. In my paper in *M.N.*, **84**, 747, 1924, a few reasons were given for the opinion that the measured Doppler shifts of the lines are due either to motions in the non-relativistic sense or to motions and certain effects consequent to the general theory of relativity. Examining the existing set of radial velocities we find them exhibiting the following general features :—

As shown in fig. 1, there is not, as formerly has been supposed, a correlation between the inclination of the plane of the spirals and the radial velocity. Thus the spirals do not move like a discus thrown through space. Neither is there any definite correlation between the radial velocity and the galactic latitude. It is true that close to the galactic poles only high velocities are to be found, but, on the other hand, a velocity of a certain size does not tell anything for certain concerning the position of the object with regard to the galaxy.

* Mount Wilson Contr., No. 243, 1922.

A rather definite correlation is shown between apparent dimensions and radial velocity, in the sense that the smaller and presumably more distant spirals have the higher space-velocity. Another interesting feature is the relation between the radial velocities and the stage of nebular evolution. If we assume an evolutionary sequence in which the globular and elliptical objects without any traceable spiral structure are the youngest, and the irregular objects, tentatively termed the Magellanic Cloud type, are the oldest, we find :—

Class of Objects.	Mean Rad. Vel. km./sec.	Number of Objects.
"Globular" nebulæ	727	11
"Early" spirals	647	18
"Late" spirals	396	5
Magellanic cloud-nebulæ *	263	4
Magellanic clouds	217	2

The most characteristic feature of the radial velocities of spirals is the presence of a very large Campbell shift of the same nature as is found in most classes of giant stars. At first it was thought that the shift shown in the spectra of spirals was constant, and the size of the K-term first determined by Wirtz and myself was fixed at +650 km./sec. Later on I reached the conclusion that the radial velocities are best represented by the expression :

$$X \cos \alpha \sin \delta + Y \sin \alpha \cos \delta + Z \sin \delta + k + lr + mr^2 - v = 0,$$

where X, Y, Z are the components of the Sun's motion relative to the spirals, k, l, and m are constants, v the observed radial velocity, and r the relative distance of a spiral derived from the apparent diameter and the total apparent magnitude, assuming the absolute dimensions to be the same.

The solutions showed that the value of m could not be very accurately determined from the 44 velocities. The final value was obtained by expressing the other unknowns as linear functions of m, and substituting these in the 44 equations of condition. Solving these again, we found the following value for the Campbell shift :

$$+513 + 10{\cdot}365r - 0{\cdot}047r^2 \text{ km./sec.}$$

The unit for r is the distance of the Andromeda nebula. It is thought that m, although inaccurately known, still expresses a real phenomenon. According to the above expression the shift reaches its maximum value, 2250 km./sec., at some 110 Andromeda units, which, according to results given later on, corresponds to a distance of 10^8 light-years. As the peculiar velocities of the spirals seems to be smaller than 800 km./sec., one would scarcely expect to find any radial velocity larger than 3000 km./sec. among the spirals.

* *N.G.C.* 6822, having a rad. vel. of +25 km./sec., should be included in this group. The following determinations of the apex were made before the velocity was published.

TABLE I.

	Material Used.	No. of Objects.	X.	Y.	Z.	V. in km./sec.	K. in km./sec.	A.	D.
								°	°
1.	All objects	44	−132	+143	− 628	− 657	+ 765	313	+73
2.	„ „ available in 1919	18	−218	−101	− 698	− 738	+ 659	25	+71
3.	Excluding N.G.C. 604 and 5195	42	−104	+135	− 620	− 643	+ 786	308	+74
4.	Irregular objects excluded	39	−113	+127	− 598	− 621	+ 786	311	+74
5.	Objects having spiral structure	26	−139	+294	− 466	− 568	+ 688	295	+55
6.	Globular nebulæ	13	−172	+480	−1092	−1203	+ 911	290	+65
7.	Irregular objects+Mag. clouds	5	−154	+358	− 10	− 389	+ 166	292	+ 11
8.	Objects in the southern galactic hemisphere	12	−935	−921	−2421	−2754	+2387	44	+61
9.	Objects in the northern galactic hemisphere	32	−645	+103	− 79	− 657	+ 146	352	+ 7
10.	Objects having large diameters	13	−594	+ 64	− 104	− 606	+ 155	353	+10
11.	Objects having small dimensions	31	− 60	+ 23	− 798	− 800	+ 933	339	+86
12.	Objects uniformly distributed	9	+ 36	−245	−1663	−1681	+1202	277	+81
13.	Nebulæ measured for rotation	5	−315	−387	− 400	− 646	+ 320	51	+39
14.	Globular clusters	18	− 91	+125	− 263	− 305	+ 31	306	+60
15.	Globular clusters	18	− 93	+ 94	− 274	− 304	0	314	+64
16.	All objects with $K=+611+4{\cdot}75r-0{\cdot}0065r^2$	44	−132	+ 21	− 628	− 642	var.	351	+78
17.	$+10{\cdot}127r-0{\cdot}025r^2$	44	−325	+415	− 79	− 533	„	308	+79
18.	$513+10{\cdot}365r-0{\cdot}047r^2$	44	−121	+190	− 637	− 676	„	302	+71
19.	$625+3{\cdot}89r$	44	−136	− 6	− 626	− 641	„	3	+78
20.	$+6{\cdot}96r$	44	−356	+358	− 11	− 505	„	315	+ 1

Table I. gives the results from the different solutions for the apex. The following small table shows the frequency of the velocities obtained after substitution in the equations :—

Limits of Velocity. km./sec.	Number of Objects. *	**	***
−1000 to −1200	..	..	1
− 600 „ − 800	1	..	1
− 400 „ − 600	1	3	5
200 „ − 400	10	8	6
0 „ − 200	11	12	8
0 „ + 200	10	9	12
+ 200 „ + 400	7	5	4
+ 400 „ + 600	3	7	5
+ 600 „ + 800	1	..	2

* Campbell shift $=513+10{\cdot}365r-0{\cdot}047r^2$ km.

** „ „ $=625+3{\cdot}89r$.

*** „ „ $=765$.

The apex for the spirals seems now to be fairly well established. In fact, the velocity and the direction have changed very little from the time when only 18 objects, or scarcely more than a third of the present material, were available. The motion of our Sun is directed towards the galactic longitude $75°\pm17°$, or nearly at right-angles to the direction (gal. long. $330°\pm5°$) where modern conceptions of the universe place the centre of our galactic system. As I pointed out before,* this may indicate a rotation of our local cluster in the plane of the Milky Way around the Sagittarius region. Assuming circular orbits, the period of rotation comes out at 3.10^9 years, which is fairly close to the value 10^9 assigned by Charlier from theoretical considerations. On the other hand, if my hypothesis is correct, the mass of the stellar system within the orbit of the local cluster will be $10^{12}\odot$, which seems to be too high a value, as the number of visible stars is 10^{10}† and the total mass of the visible stellar universe something about $0{\cdot}8.10^{10}\odot$. Of course, dark stars and dark matter exist and increase that value, and, besides, the assumed distance for the central region (68,000 light years according to Shapley) may be somewhat too large. Still it seems likely that the apparent motion of the Sun may not be entirely attributed to a rotation of the local system.

As to the proper motions of the spirals our present knowledge is not very satisfactory. Many of the spiral nebulæ and the objects related to them have a very definite nucleus permitting accurate settings. Time after time measures have been performed and catalogues worked out giving accurate nebular positions. The oldest measures that can be used advantageously are from 1860-1865. The most accurate positions for this epoch were determined by Professor H. Schultz in Upsala, who measured 500 nebular objects. His results are very accurate and remarkably free from systematic errors.

Inasmuch as several astronomers during the last few years have advocated very moderate distances for spiral nebulæ, it occurred to me that it may be worth while to try to derive proper motions from the material existing. For 100 objects belonging to what may be called the family of spirals, about 1200 positions have been collected, using the following catalogues :—

Laguier, d'Arrest (I. and II.), Auwers, Schönfeld (I.–II.*a*), Harvard, Lord Rosse, Schmidt, Oppolzer, G. Rümker (I. and II.), Vogel (I. and II.), Schultz (I. and II.), Engelmann, Stephan (new reduction by Esmiol), Kempf, Porter (I. and II.), Weinek and Gruss, Ginzel, d'Engelhardt, Stone, L. Becker, Dreyer, Mönnichmeyer, Spitaler, Bigourdan, Merecki, Keeler, Wolf, Lorentz, Rheinmuth (I. and II.), Kobold, Wirtz, Winnecke (new reduction by E. Becker), Bredichin, Peters, and occasionally positions given in other sources, for instance, some of the star catalogues.

A number of the older positions have been improved by Wirtz or myself using new positions for the comparison stars.

The nebular positions have been corrected for systematic errors and weights assigned, whereby a compromise was made according to

* *M.N.*, **84**, 747, 1924.
† *Nature*, No. 2903, 1925 June 20.

the following factors : (1) the number of observations and the internal agreement ; (2) the brightness and the accuracy of the positions of the comparison stars ; (3) the method of observation (micrometer, heliometer, meridian instruments, photography) ; (4) the instrument used ; (5) the size of the systematic errors.

The weights run from 0–6, the last given only to a few modern photographic measures. Of the old positions those of Schultz have, as a rule, the weight of 4.

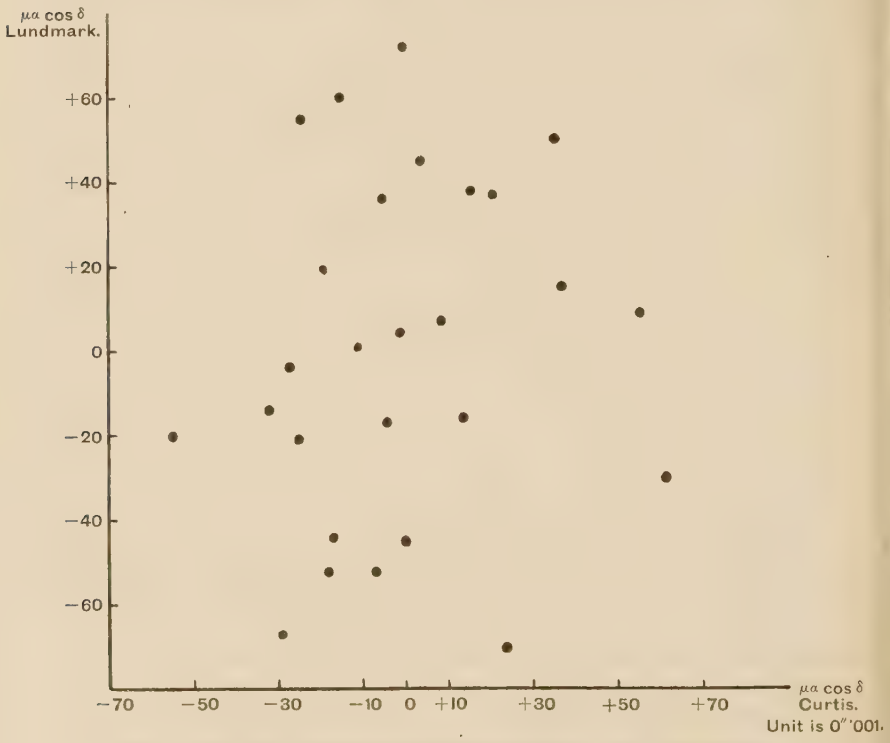

FIG. 2.—Comparison between the Right-Ascension Components of the Proper Motion of Spiral Nebulæ as derived by Curtis and by Lundmark.

Through the efficient aid of Mr. K. Littmarck, M.A., least square solutions have been carried through for 83 objects. The average values of the proper motions are : $\bar{\mu}_\alpha$ $-0^s\cdot0028$; $\bar{\mu}_\delta=0''\cdot031$, and the algebraic means, $\mu_\alpha=-0^s\cdot0000$, $\mu_\delta=+0''\cdot013$; the average mean error is $\epsilon(\alpha)=\pm0^s\cdot0037$; $\epsilon(\delta)=\pm0''\cdot039$. The average value of the total proper motions is $0''\cdot045$. The motions derived are evidently of the same order of magnitude as their errors.

Computing the apex of the Sun's way from my results I have found : $A=19^h\cdot2$; $D=-69°$; $q=0''\cdot0135$. The poor agreement with the apices derived from radial velocities shows that some systematic error must have crept into the material.

A comparison of my proper motions with those of Curtis will strengthen this conclusion. Measuring Crossley plates with a mean interval of 13·85 years, Curtis found, as is well known, a mean motion for 66 spirals of 0″·033. Using his values according to Trümpler's rediscussion we find the apex: $A=4^h\cdot0$; $D=-59°$; $q=0''\cdot0147$.

Twenty-nine objects have been determined both by Curtis and by myself. Comparing them we find no systematic difference in declination, but for the right ascension components there is the relation:

$$\mu_L-0''\cdot013=\mu_C,$$

where the subscripts denote the motions of the observers. In the solution made for the apex, using my material, the μ_L have accordingly been reduced to Curtis' system.

As will be seen in the graphs, there is practically no relation between the proper motions of Curtis and myself. The correlation-coefficients are, in fact, illusory: $r=-0\cdot023\pm0\cdot192$ (for the right ascensions), and $r=+0\cdot147\pm0\cdot181$ for the declinations).

Assuming the real motions distributed as accidental errors, we have the following relations:

$$\begin{aligned}(n-1)\ (\mu^2+\epsilon_L{}^2)&=\Sigma\mu_L{}^2\\(n-1)\ (\mu_2+\epsilon_C{}^2)&=\Sigma\mu_C{}^2\\(n-1)\ (\epsilon_L{}^2+\epsilon_C{}^2)&=\Sigma(\mu_L-\mu_C)^2,\end{aligned}$$

where ϵ are mean errors, μ the mean of the squares of the individual motions, and n the number of objects in common. From the material we get the following results:

$$\begin{aligned}\epsilon_L(\alpha)&=\pm0''\cdot044 & \epsilon_L(\delta)&=\pm0''\cdot033\\\epsilon_C(\alpha)&=\pm0''\cdot033 & \epsilon_C(\delta)&=\pm0''\cdot024\\\mu_\alpha&=\pm0''\cdot018 & \mu_\delta&=\pm0''\cdot016.\end{aligned}$$

Thus there should be an equal chance that the real proper motions in right ascension are inside the limits $\pm0''\cdot012$. The corresponding limits for the declination components are $\pm0''\cdot011$. It is not likely that the motions actually are as large as 0″·01, but the computation shows that Curtis' motions as well as my own are probably spurious and mainly due to the errors of the methods.

Another set of proper motions of nebulæ is obtained by van Maanen as a by-product from his measures of internal nebular motions. The outcome of these valuable proper motions has been discussed in the *Observatory*, **57**, 279, 1924. From the objects in common with Curtis, we find:

$$\begin{aligned}\mu_C(\alpha)-0''\cdot006&=\mu_{vM}(\alpha)\\\mu_C(\delta)-0''\cdot025&=\mu_{vM}(\delta).\end{aligned}$$

Although the systematic differences are uncertain on account of the few objects in common, a new solution for the apex has been made for Curtis' proper motions reduced to the system of van Maanen. The results are given in the table below.

Now, it may very well be possible that the proper motions derived by me are affected by an error in the adopted value of the constant

of precession. The spiral nebulæ in any case are rather distant objects, but the precession is computed using values derived from neighbouring stars. In order to test the question the following equations were formed :

$$x \sin \alpha - y \cos \alpha + \Delta n \sin \alpha \sin \delta + \Delta \mu_0 \cos \delta = \mu_\alpha \cos \delta$$
$$x \cos \alpha \sin \delta + y \sin \alpha \sin \delta - z \cos \delta + \Delta n \cos \alpha = \mu_\delta,$$

where x, y, z are the components of the Sun's motion, $\Delta \mu_0$ a constant

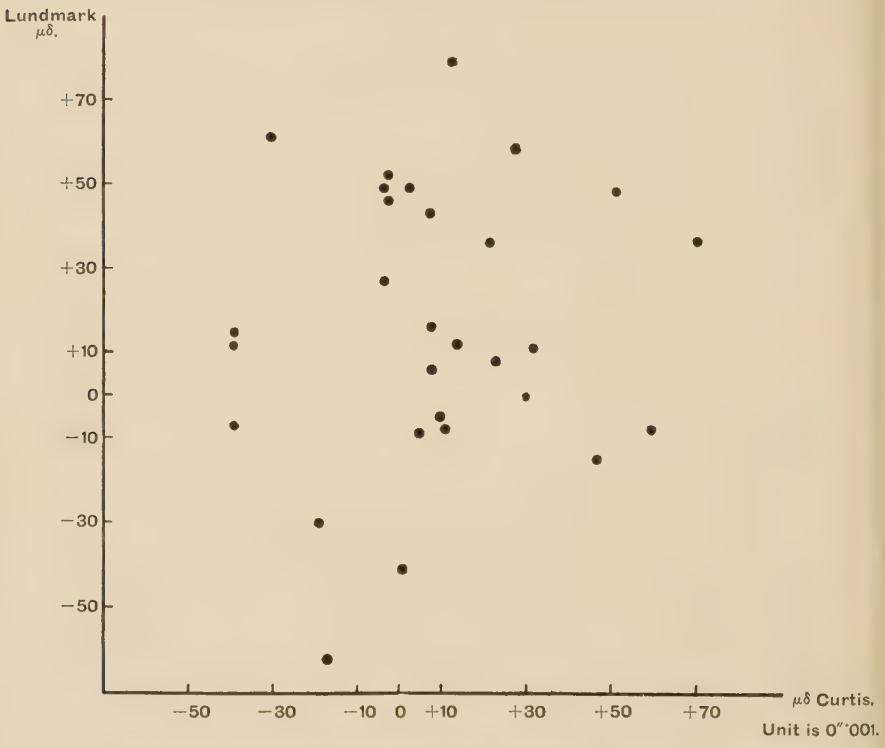

FIG. 3.—Comparison between the Declination Components of Proper Motion as derived by Curtis and by Lundmark.

correction, and Δn the correction to the precessional constant, n, the annual motion of the celestial pole towards the equinox.

The value of $\Delta \mu_0$ is $=-0''{\cdot}0096$ and the value of Δn is $=-0''{\cdot}015$. The latter value should be the correction to the constant of precession used, which is Struve-Peters for 1900.* It seems to me rather doubtful if the correction is real, as Boss derives a positive correction and so does Eichelberger. Certainly, it will be desirable to wait for a new determination of Δn when more material has been collected.

* The correction to Newcomb's constant would be $-0''{\cdot}012$, which is very improbable. The correction found must certainly be attributed to other causes than an error in the precession constant.

As to the motions of Curtis or van Maanen, they are not affected by any influence of precession.

It is interesting to note that the apex derived from the above equations is rather close to the antapex of ordinary stars, and this is also the case with the apex computed from Curtis' uncorrected motions. It can easily be shown that if what we measure is not the motions of the nebulæ, but the reflected motions of the comparison stars, this will give us the co-ordinates of the antapex when we make a solution for the apex.

In the following table we give a summary of the determinations of apices from the proper motions of spirals:

Material.	No. of Objects.	x.	y.	z.	q.	A.	D.
		″	″	″	″	h	°
Curtis' motions	66	+0·0056	+0·0038	−0·0132	0·0147	4·0	−59
„ „ corrected	66	+0·0067	+0·0026	+0·0154	0·0170	1·4	+65
Lundmark's motions corrected	83	+0·0016	−0·0048	−0·0145	0·0153	19·2	−69
Lundmark's motions corrected and a correction to the precession constant supposed	83	+0·0108	+0·0062	−0·0109	0·0165	2·0	−41
Van Maanen's motion	7	+0·0035	+0·0025	+0·0041	0·059	2·4	+44
Van Maanen's motion rediscussed by Smart	7	+0·0030	+0·0014	+0·0064	0·0074	1·7	+63
Wirtz's motions *	98	+0·0089	+0·0248	+0·0178	0·0320	4·7	+34

Although the proper motions are not as yet known, still we can get an idea of the mean parallax of spiral nebulæ. Our discussion shows that the real motions of spirals are well below 0″·01. Using the values of q as representing an upper limit for the real q and combining with the determinations of the apex from radial velocities, we find a minimum distance of 30,000 light-years for the spiral nebulæ.

To this will only be added that the dropping of the well-established K-term will give a mean distance of the same order of magnitude, because the assumption K=0 leads to a value for the velocity of the Sun of the same size or even larger than the one adopted here.

(c) *Internal Motions.*—If the radial velocity can be measured for different points in the same spiral, and measures of direct photographs have given the apparent size of internal motions, fairly reliable distances can be derived from such data. As is well known, Dr. van Maanen has measured the internal motions of 7 spiral nebulæ. During a stay at the Mount Wilson Observatory, at the suggestion of Dr. Adams and Dr. van Maanen, I measured the same plates of Messier 33 as the latter. The same instrument, the new large stereocomparator with a "blink" arrangement, was used for the measures. Altogether 400 points well distributed in the spiral arms, referred to 24 comparison stars, were measured. The measures were made at the same time as van Maanen

* Wirtz's values, which have not been published in detail, evidently are not computed from all the existing material for every object but from three or four positions in the average. Regarding details, see *A.N.*, **204**, 26, as well as other papers by Wirtz, giving the results of his extensive investigations concerning proper motions of different groups of nebulæ.

performed his work, but the reduction of my set was carried out after my return to Upsala. The detailed comparison of the two series of measures will be given later on, but the following general results may be mentioned.

The proper motion of the nebula as a whole is found to be: $\mu_\alpha = -0''.0015$; $\mu_\delta = -0''.0050$, whereas van Maanen has $\mu_\alpha = +0''.0034$; $\mu_\delta = -0''.0044$. Resolving my motions into rotational and radial components, the mean values are found to be: $\bar{\mu}_{rot} = +0''.0016$; $\bar{\mu}_{rad} = -0''.0056$. The corresponding values from van Maanen's measures are: $+0''.0196$ and $-0''.0029$ respectively. The dispersions around the means are: $\sigma_{rot} = \pm 0''.0065$; $\sigma_{rad} = \pm 0''.0091$ (Lundmark); $\sigma_{rot} = \pm 0''.0080$; $\sigma_{rad} = \pm 0''.0096$ (van Maanen).

The rotational motion found by me can be represented by the linear formula:

$$\mu_{rot} = -0''.00084 + 0''.00032\rho,$$

where ρ is the distance from the centre of the nebula in minutes of arc.

A similar solution of van Maanen's components gave:

$$\mu_{rot} = +0''.01063 + 0''.00119\rho.$$

As the mean distance from the centre is $7'.61$, the mean period of rotation will be $2.8.10^6$ years and $0.23.10^6$ years respectively.

If we use van Maanen's results, comparing them with the rotation of the spirals N.G.C. 224, 3031, and 4594, spectroscopically measured, we find distances ranging from 10,000 to 40,000 light-years. My measures suggest distances between the values 40,000–160,000 light-years. On account of the few cases investigated as yet, considerable uncertainty is involved in the application of the method.

The following formula is easily deduced:

$$M = [3.71856]\mu_{rot}^2 \tan \rho \,.\, \pi^{-3},*$$

where M is the mass within the distance ρ from the nucleus of the nebula expressed in minutes of arc, and π the parallax of the nebula. Assuming a parallax of $0''.000003$ we find the following values for M:

$1.6 \,.\, 10^{14} \odot$ (van Maanen)
$1.1 \,.\, 10^{12} \odot$ (Lundmark).

(d) *Absorption in Space.*—Another direct way for estimating the spiral distances could be obtained from determinations of the amount of selective absorption in space. The effective wave-lengths of spirals as measured in Upsala by Lindblad and myself † permitted a computation of the spectral types by using the relation between λ_{eff} and spectra as established from investigations of stars. The graph in fig. 6 shows the fair agreement between the computed and classified spectra. Now, the former depend entirely on the energy distribution in the continuous spectrum, whereas the latter are obtained from scrutinising the balance between certain groups of absorption lines. Thus a selective absorption, if present, will make the computed types later than the observed ones. The mean spectral type computed is G4·5 and the mean observed G1·7. Evaluating the spectral types in colour-index we find

* Publ. *A.S.P.*, **34**, 108, 1922. † Aph., J., **46**, 206, 1917; **50**, 376, 1919.

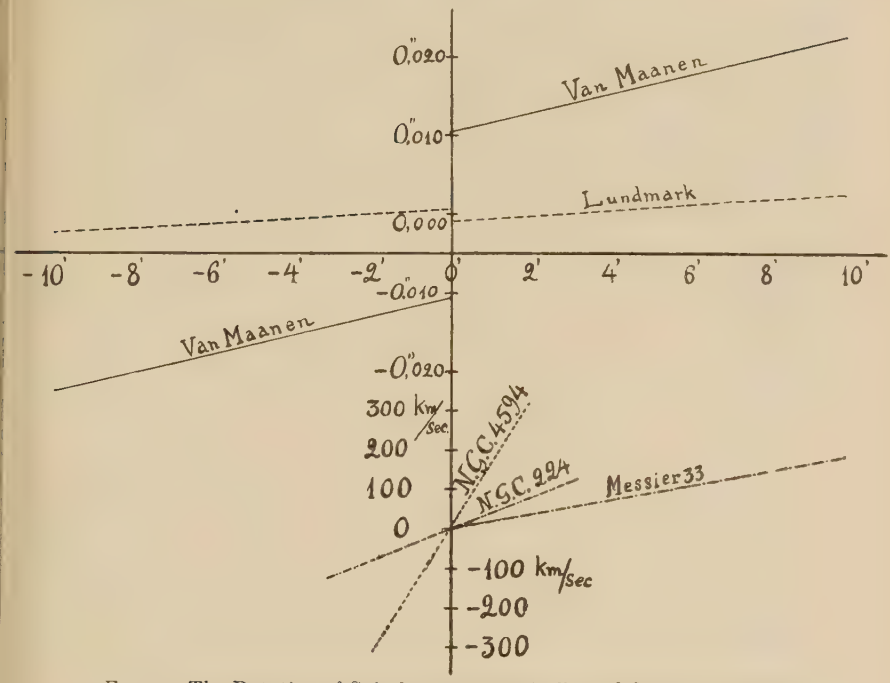

FIG. 4.—The Rotation of Spirals spectroscopically and directly measured. Abscissæ are angular distance from the centre of the Spirals.

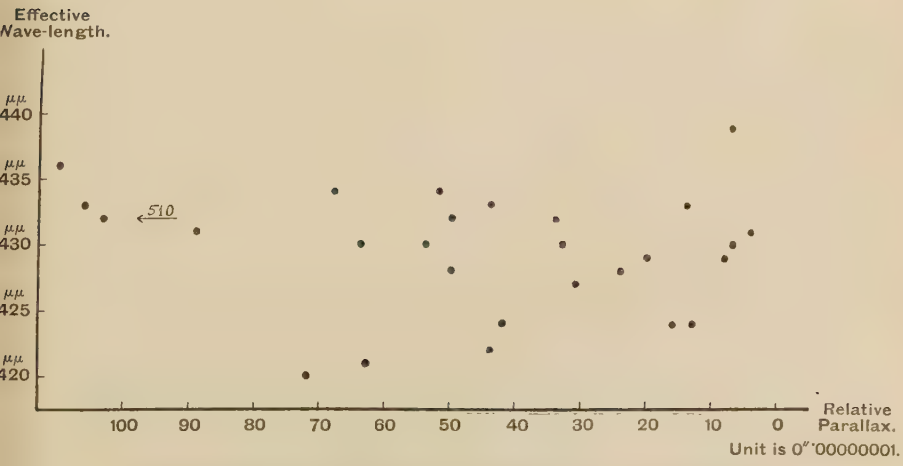

FIG. 5.—Effective Wave-Lengths of Spiral Nebulæ as plotted against relative Parallaxes of the Objects.

the difference to be $+0^{m}\cdot12$. If we had a final determination of the coefficient of the selective absorption we could derive *a maximum value* of the mean distance of the spirals. As the value of Shapley is a limiting value, showing that the coefficient is smaller than $0^{m}\cdot000002$ per parsec, I prefer for the present to go the other way and make an independent estimate of this coefficient. The mean distance of the objects used is 20 Andromeda-units, which should correspond to a distance of 20 million light-years. To be on the safe side, I assume 10

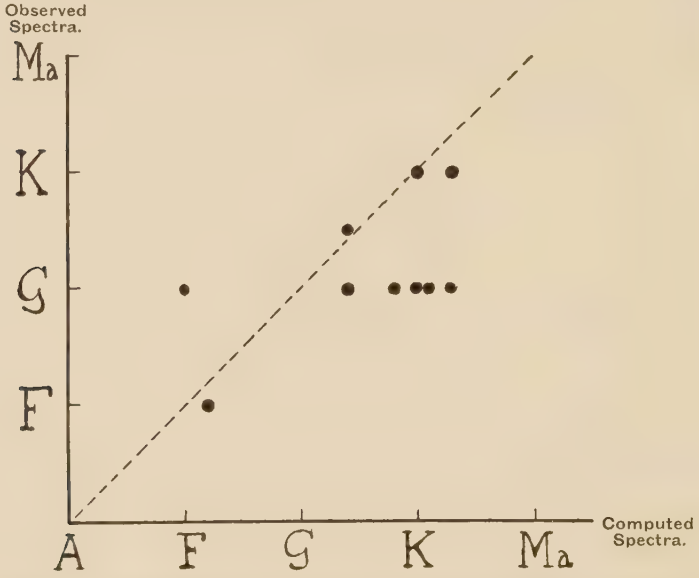

FIG. 6.—Comparison between Spectra of Spirals computed from effective Wave-Lengths and directly classified.

million light-years as representing the mean distance, and thus we find the following value of the coefficient of selective absorption :

$$\leq 0^{m}\cdot000000012 \text{ per light-year.}$$

The interesting question arises : Is there any general absorption among the spiral nebulæ ? The graph in fig. 7 shows a comparison between the photometric measures of the surface intensity of spiral nebulæ measured by Wirtz and the relative parallaxes computed by me. As pointed out by Wirtz himself, if there are not systematic errors making the smaller spirals measured fainter than the larger ones, the only possible explanation will be that we are dealing with a general absorption or rather extinction affecting equally all wave-lengths. From the material used in the graph I have derived a coefficient of the absorption amounting to $0^{m}\cdot02$ per Andromeda-unit. Assuming as before 10^{6} light-years for that unit, we have :

$$\text{general absorption} = 0^{m}\cdot00000002 \text{ per light-year.}$$

At the suggestion of Professor Hertzsprung the density of meteoric matter has been computed from this coefficient. Using the formula : *

$$\text{extinction in magnitudes} = \frac{p \cdot m}{d \cdot s},$$

where p is a constant $=0{\cdot}407$, m the average meteoric matter in unit space, and d the mean diameter of meteors, and s their mean density, we have adopted the following values : $d=6{\cdot}25 \cdot 10^{-3}$ grams, † $s=7{\cdot}9$

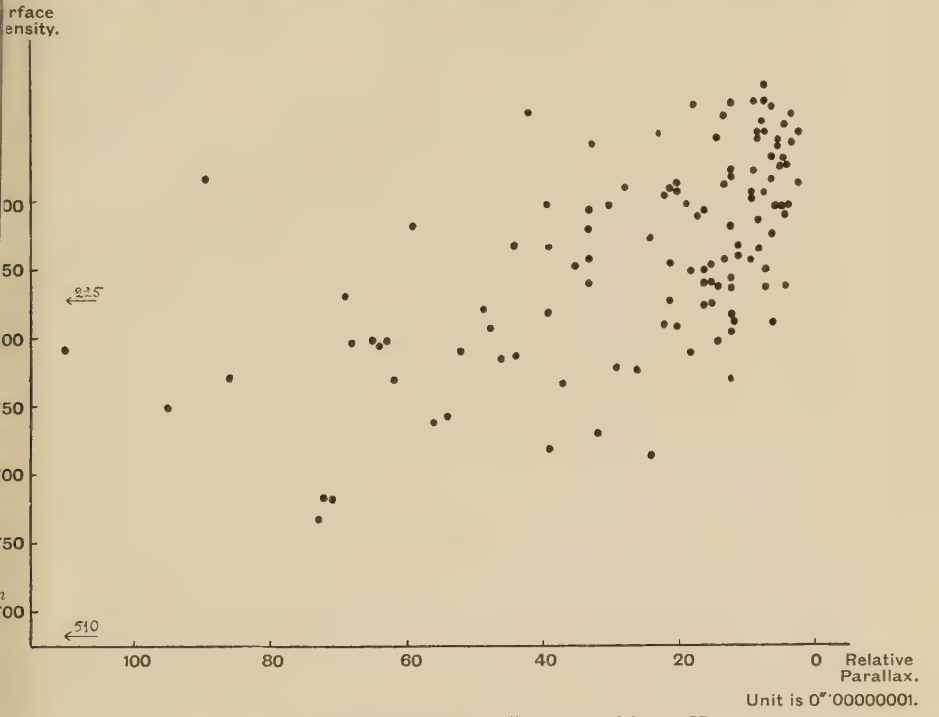

Fig. 7.—Relation between the Surface Intensity of Spiral Nebulæ and the relative Parallax.

(iron). Thus we find that within a cube having a side of one astronomical unit there should be :

$$0{\cdot}6 \cdot 10^{16} \text{ meteors.}$$

According to Hoffmeister's investigations,‡ founded on actual observations of the frequency of meteors, the number in the same cube is :

$$10^{16} \text{ meteors.}$$

The excellent agreement strengthens the conclusion that Wirtz's observations really show the presence of a general absorption in space.

* *Proc. of the Nat. Academy of Sc. Washington*, **8**, 115, 1922.
† *Proc. of the R.S.*, **102**, 411, 1922.
‡ *Ergänzungshefte zu den Astr. Nachr.*, Bd. 4, Nr. 5, 1922.

B. Indirect Methods.

(a) *Novæ in Spirals.*—One of the most interesting characteristics of the spiral nebulæ is perhaps that they seem to possess the necessary conditions for producing the so-called Novæ, as also is the case with certain regions in our stellar system, as a rule situated close to the galactic plane. The first Nova discovered in a spiral nebula was the one named S Andromedæ which appeared in 1885 close to the nucleus of the great nebula in Andromeda. As this spiral covers an area of 1/42,000 of the sky, the probability is rather small that we should have seen the star only by chance in direction towards the nebula. At the time of discovery several astronomers thought this was the case, and in spite of a few subsequent appearances of Novæ in spirals or related objects (N.G.C. 5253, 6093, 4147, and 5457–58) the importance of the phenomenon was not realised till later on. In the year 1917 it was concluded from the discoveries of several American astronomers that the appearance of Novæ in spirals must be a rather common event. Up to the present 22 Novæ have been observed in the Andromeda nebula.* The probability that these objects are seen only by chance in the same direction as the nebula is of the order of 4.10^{-88}, and thus there is not the slightest doubt that the stars in question actually are situated in the spiral. If any other proof should be thought necessary, it may be pointed out that there is a distinct crowding of Novæ around the central parts of the Andromeda nebula. Besides, as Novæ are not known in the stellar regions in the neighbourhood of the nebula, the discoveries are not favoured by a more thorough scrutiny of this spiral than of other parts of the sky.

Curtis † seems to have been the first to realise the cosmical importance of the occurrence of Novæ in spiral nebulæ. By assuming equality in absolute magnitude for galactic and spiral Novæ he concluded that the latter, being apparently ten magnitudes fainter, are about one hundred times as far away as are the former. Estimating the mean distance of galactic Novæ to be 100,000 light-years, he reached the conclusion that the spirals are galactic systems in size comparable with our own, and *that the closest spirals are millions of light-years away.* In 1919 I discussed the different ways of estimating the distances of spirals.‡ In this work the mean absolute magnitude of four galactic Novæ was derived as −4. From that value the parallax of the Andromeda nebula was computed to be 0″·0000051 (650,000 light-years) and the dimensions 23,000 light-years. That value has been accepted or confirmed later on by Curtis, § Luplau-Janssen, Haarh, ‖ Doig, ¶ and others on the basis of Novæ, and by Oepik ** and Hubble †† on the basis of rotational evidence and appearance of Cepheids respectively.

* In fact, 24 more Novæ have been discovered by Mr. Hubble, but particulars have not been published as yet.

† *Journal of the Washington Acad. of Sc.*, **9**, No. 8, 1919; *L.O.B.*, 300.

‡ *Astr. Nachr.*, **209**, 369, 1919; *Kungl. Sv. Vet. Akad. Handl.*, Bd. 60, 1920.

§ *Rivista di Scienza*, **35**, No. 1, 1924.

‖ *Astr. Nachr.*, **215**, 285, 1922.

¶ *Journ. of the B.A.A.*, **32**, 138, 1922.

** *Ap. Journ.*, **55**, 409, 1922.

†† *Observatory*, **48**, 139, 1925.

Fig. 8.—Galactic Distribution of Novæ (those in Spiral Nebulæ are excluded).

During the following years the absolute magnitudes of the galactic Novæ have been determined by me, using different methods. In the following I give a brief summary of the results.

a. Direct Measured Parallaxes or Determinations of Hypothetical Parallaxes for Individual Objects.—The following table gives a summary of the results:

Object.	Mean of direct Measured Parallaxes.	Proper Motion Parallax.	Spectral. Parallax	Hypothetical Parallax.	Adopted Mean.	Absolute Magnitude. At Maximum.	Absolute Magnitude. At Minimum.
Nova Persei 2	″0·0082	″0·0070	″..	″0·0093	″0·0082	−5·4	+8 to +1
„ Aquilæ 3	0·0025	0·0026	0·0022	0·0040	0·0031	−8·8	+3·0
„ T Coronæ	..	0·0016	0·0015	..	0·0016	−7·1	+0·6
„ Geminorum 2	0·005	0·0080	..	..	0·0060	−2·8	+7·0
„ Lacertæ	0·006	0·0003	..	..	0·0040	−1·1	+8·4
„ RT Serpentis	..	..	0·003	..	0·0030	?	+7·4
„ T Scorpii	..	..	..	0·00006	0·00006	−9·1	?
„ P Cygni	−0·013	0·0058	0·0032	..	0·0021	−4·6	−2·7
„ T Aurigæ	..	0·0015	..	..	0·0015	−4·6	+5·0
„ η Carinæ	..	0·0017	..	0·0015	0·0016	−7·0	+0·6
„ Cygni 1920	0·010						
„ Ophiuchi No. 4	−0·005						

Remarks.—*Nova Persei*—It will be of interest to see how the agreement is between different measures of the parallaxes of this star:

$\pi_{abs.}$	$\pi_{abs.\ red.}$	Weight.	Authority.
$+0''\cdot026\pm0''\cdot010$	$+0''\cdot032$	0·8	Bergstrand.
$+0''\cdot020\pm0''\cdot020$	$+0''\cdot025$	0·3	Hassenstein.
$+0''\cdot005\pm0''\cdot004$	$+0''\cdot005$	2·0	Küstner.
$-0''\cdot012\pm0''\cdot023$	$-0''\cdot001$	0·5	Chase.
$-0''\cdot010$	$-0''\cdot010$	0·3	Hartwig.
$+0''\cdot027\pm0''\cdot009$	$+0''\cdot027$	0·3	Rambaut.
$+0''\cdot007\pm0''\cdot004$	$+0''\cdot006$	2·0	Van Maanen.

The second column gives the parallax reduced to absolute parallax and to Van Maanen's system.

The parallaxes in the upper table headed "proper motion parallax" are derived from the relation between the quantities m_π and m_μ as established for different spectral classes.* The spectral parallaxes have been derived from the knowledge of the spectrum when the star was in its normal condition.†

Various hypothetical parallaxes have been derived for this star using the reflection theory, that is the assumption that the outburst was reflected from the dark matter around the star. Of the different values we have selected the one given by Turner.‡ The close agreement between the values must be taken as a strong argument in favour of the reality of the derived parallax.

* *Lick Obs. Bull.*, No. 339, 345, 1922.
† *Publ. A.S.P.*, **34**, 147, 1922.
‡ *M.N.*, **79**, 23, 1918.

Nova Aquilæ No. 3.—The following parallâxes have been measured :—

	$\pi_{abs.}$	$\pi_{abs.\ red.}$	Weight.	Authority.
Photographic determinations	+0″·019±0″·006	0″·018	1·5	Van Maanen.
	−0·001±0·008	0·007	1·2	Daniel.
	−0·011±0·005	−0·007	1·0	Alden.
	−0·008±0·007	−0·004	1·0	Olivier.
	−0·004±0·013	−0·004	0·8	Fox.
	−0·014±0·006	−0·006	1·0	Lee.
From Meridian observations	+0·071±0·013		0·3	Courvoisier.
	+0·197±0·050	(−0·041)	0·2	Philippot.
	+0·064±0·072		0·2	Delporte.
	+0·160±		0·1	Trousset.

In *Publ. A.S.P.*, **34**, 208, the value of Olivier was erroneously quoted by me, and the parallax of Alden had been overlooked.

The meridian observations, which are evidently influenced by systematic errors, could just as well be excluded in forming the mean. As shown by Henroteau, the large positive parallax by Philippot can be reduced to a negative one when the proper motion is introduced into the solution. As the actual proper motion is certainly very small, the large value resulting from Philippot's material has to be interpreted rather as a magnitude equation varying with the time.

The hypothetical parallax value was derived by me by comparing the expansion in angular and linear speed of the nebulous disk.*

Nova T Coronæ.—My spectrograms from 1921 showed the star to be an Mb giant, from which fact the spectral parallax was derived.

Nova Geminorum No. 2.—Two directly measured parallaxes have been published, viz., +0″·006±0″·008 by Slocum, and −0″·019±0″·013 by Miller.

Nova Lacertæ.—The measured parallaxes are, +0·″007±0″·012 by Slocum and Mitchell, and −0″·005±0″·020 by Balanowsky.

Nova RT Serpentis.—The parallax is, in fact, a "spectroscopic" one, taken from Mount Wilson Contr., No. 199.

Nova T Scorpii.—This Nova is of unusual interest as having appeared close to the centre of the globular cluster Messier 80. As pointed out before, it probably is a *bona fide* Nova, really situated in the cluster. Using Shapley's parallax for M80, we obtain the absolute maximum magnitude −9·5, and using my value −8·7.

Nova P Cygni.—Parallax measured by Slocum and Mitchell, −0″·021±0″·016.

Nova η Carinæ.—For the nebula itself a parallax of +0″·0015 has been deduced in *Publ. A.S.P.*, **34**, 40, 1922. I have assumed the Nova to be connected with the nebula, which gets confirmation from the value of the proper-motion parallax.

Nova Cygni 1920.—Two trigonometric values for the parallax have been published, viz., 0″·020±0″·006 (Daniel),† and 0″·003±0·″005 (Van Maanen).‡

Nova Ophiuchi No. 4.—Van Maanen has measured the parallax −0″·005±0″·008.§

Using all the measured parallaxes alone (positive and negative values), the mean absolute maximum magnitude is found to be −7·0. If we combine the different hypothetical parallaxes together with the measured we get −6·2 for the same quantity.

β. Proper Motions and Radial Velocities.—From a discussion of the 11 proper motions given in *Publ. A.S.P.*, **34**, 207, and the 8 radial velocities quoted in the same paper, a mean absolute magnitude of −7·7 at maximum has been derived. One may think that the proper

* *Publ. A.S.P.*, **33**, 218, 1921.
‡ *A.J.*, **36**, 46, 1925.
† *Pop. Astr.*, **32**, 221, 1924.
§ *Ibid.*

motions are too small and inaccurate to allow a derivation of mean parallax. It may be very instructive to compute the values of the quantities $m_\pi = m + 5 + 5 \log \pi$ and $m_\mu = m + 5 + 5 \log \mu$ from the Nova material. In fig. 9 the relation between the absolute magnitude and the absolute proper motion magnitude is given from the best parallaxes and proper motions we have for the present. The fact that the same quantities for Novæ at minimum light fall rather nicely into the diagram suggests that the parallaxes of Novæ as well as their proper motions are free from systematic errors of importance.

γ. *Dip Effect of the Novæ.*—The Sun is situated some 200 light-years to the north of the galactic plane. The Milky Way star clouds have a mean galactic latitude of $-0°{\cdot}98$ and the Novæ have a "dip" of $-1°{\cdot}02$, showing the intimate relationship between these and the Milky Way structure. The mean distance that results from the dip gives a mean absolute magnitude at maximum of $-6{\cdot}5$.

δ. *The concentration of the Novæ in the Sagittarius region* gives another means for deriving the mean absolute magnitude. If we presume that the 16 Novæ found in the region actually belong to the central parts of the stellar system, we get, adopting Shapley's value of 68,000 light-years for the distance, the $m_\pi(\text{max.}) = -9{\cdot}5$. Correcting the value according to Wilson's determinations of the zero point in the period-luminosity relation of Cepheids we obtain $m_\pi(\text{max.}) = -8{\cdot}7$.

ϵ. *Comparison between the galactic distribution of Novæ and other classes of Milky Way objects* gives another possibility of estimating m_π for Novæ. Suppose, for instance, the mean of the co-ordinates of Novæ and B stars to be the same, or that the dispersions in the same co-ordinate are equal. This gives $m_\pi(\text{max.}) = -4{\cdot}2$.

Summing up the results from the different methods we have:

Method.	m_π (max.).	Weight.
α	$-6{\cdot}2$	3·0
β	$-6{\cdot}5$	1·0
γ	$-8{\cdot}2$	1·0
δ	$-7{\cdot}7$	3·0
ϵ	$-4{\cdot}2$	0·5
Unweighted mean	$-6{\cdot}6 \pm 0{\cdot}70$	
Weighted mean	$-6{\cdot}9 \pm 0{\cdot}66$	

The latter value will be adopted in this paper.

We see that the Novæ at their maximum are among the absolutely brightest stars known in the Galaxy. If the same is the case with the Novæ in spirals it may explain why *Cepheids* or other variable stars have not been discovered as yet.* These stars being absolutely fainter than the Novæ will only reach 18^m or 19^m at their maximum, and will not be easily discernible on the nebulous background.

I have chosen m_π (max.) as the characteristic magnitude of the

* Since this was written, Dr. Hubble has announced the discovery of Cepheids in M31, M33, M81, and N.G.C. 2403.

Novæ, because the absolute magnitude seems to be more or less constant at maximum, but varies considerably at minimum. To illustrate that statement we cite the following circumstances.

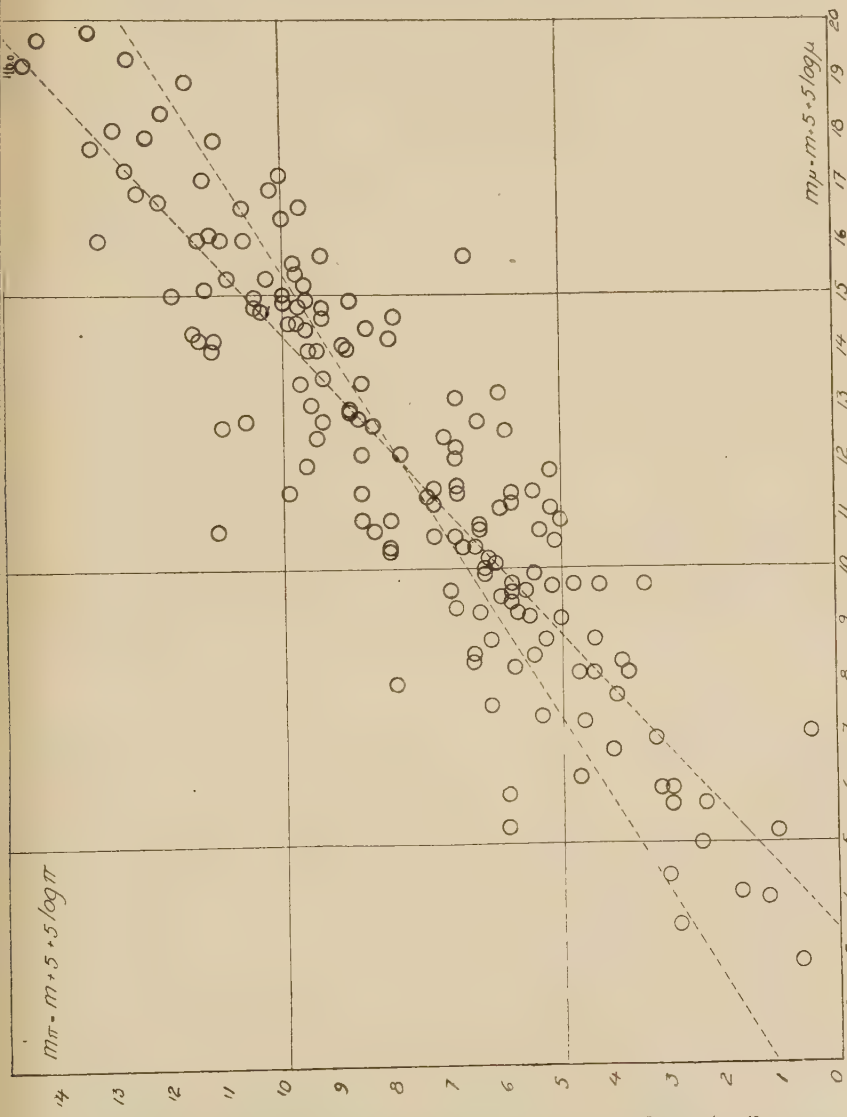

FIG. 9.—Relation between Absolute Magnitude, m_π, and Absolute Proper Motion Magnitude, m_μ, as derived from the Stars having Parallaxes larger than 0".100.

The parallaxes of the objects *Nova Persei* 2, *Nova Aquilæ* 3, and *Nova T Coronæ* have been determined by several independent methods, and are thought to be fairly accurate. We have for m_π (max.) −5·4, −8·7, −7·1, and for m_π (min.) +8·8, +3·2, and +0·6. Thus

the minimum or the original magnitude varies considerably more than does the maximum magnitude, which is also seen from the table of individual parallaxes of Novæ.

In the central part of our galactic system 16 Novæ have been found. As the region in question is very far away we can take the apparent magnitudes of the Novæ as also giving the distribution of the absolute magnitudes. Excluding Kepler's Nova, which is of a singular type, we find the dispersion about the mean to be $\pm 1{\cdot}52$ magnitudes.

The 22 Novæ observed in the Andromeda nebula have, with the exception of S Andromedæ, a fairly constant absolute magnitude. Besides, we ought to remember that it is very seldom that conditions permit us to observe an Andromeda Nova at maximum brightness, a fact that increases the dispersion in absolute magnitude. The same remark refers, of course, to the faint Novæ in the Milky Way.

Computing m_μ (max.) for 11 Novæ the dispersion is $\pm 2{\cdot}31$, whereas the dispersion in m_μ (min.) is considerably larger, $\pm 3{\cdot}31$. As we have approximately $m_\pi = a + 0{\cdot}8 m_\mu$ and $\frac{dm_\pi}{m_\pi} = 0{\cdot}8 \frac{dm_\mu}{m_\mu}$, the dispersions in m_π (max.) and m_π (min.) are $\pm 1{\cdot}78$ and $\pm 2{\cdot}56$ respectively.

How do we know that the spiral Novæ are of the same kind and have the same physical properties as galactic Novæ? Of course the co-ordination of the two groups is a hypothetical method, but not bolder than many other assumptions in modern astronomy. Besides, the hypothesis gains some support from the following circumstances:

1. The similarity of the light-variations. The light-curves obey the same general law of rapid increase and a more slow decrease. The curve derived from observations of S Andromedæ is very like the curve of a typical Nova.* Secondary oscillations having a period of three days are present, and the general course of the variations is that of galactic Novæ.

2. The rate of the decrease of light is in the mean $0^{m}{\cdot}052$ for 9 spiral Novæ sufficiently observed. In fact the decrease after the time of maximum is in the mean of the same magnitude as in the case of galactic Novæ.

3. As far as our present knowledge goes there are certain resemblances between the spectra of galactic Novæ and those of spiral Novæ; *e.g. Nova T Coronæ* and *Nova Z Centauri* had spectra of the classes Mb and R, and *Nova N.G.C.* 6946 had a spectrum crossed by what appeared to be series of bright bands. *Nova S Andromedæ* showed emission lines in a spectrum which seems to have been of a late type, etc.

4. There is a certain resemblance between the galactic Novæ and the Andromeda Novæ with regard to their distribution in the system in which they appear. In both cases the Novæ are crowded around the central parts. As pointed out in various papers,† the Novæ in our stellar system, although confined to the galactic regions, are not found in the star clouds, but at the borders of the Milky Way or in relation to the dark and bright nebulæ. It may have some significance that the Andromeda Novæ prefer that side of the nebula where evidently

* *Kungl. Sv. Akad. Handl.*, Bd. 60, v, 1920. † *Publ. A.S.P.*, **35**, 95, 19.

the absorption is rather marked. In some cases they have also appeared in small dark " rifts " of the nebula.

5. There is every reason for the view that Novæ are not *new stars*, but originate from different classes of ordinary stars. If the Andromeda nebula were constantly watched, ten Novæ or more per year would probably be discovered. Thus the number of Novæ outbursts in the nebula must be enormous as soon as we take astronomical epochs into account. The possibility remains that the same star passes through the Nova-stage time after time, but this seems to be a rare event compared with the number of outbursts. Thus it seems fair to assume that the Andromeda nebula is a vast stellar system consisting of millions of stars, and not a comparatively small organisation such as a stellar cluster.

In order to derive the distance of the nebula, I thus assume equality as to the mean absolute maximum magnitude of the two groups. Excluding S Andromedæ, the mean apparent magnitude at maximum is $17^{m}\cdot03$. Now, one must remark that the recorded magnitudes do not give the maximum brightness. The nebula can be observed only now and then, and inspection shows that most of the observed Novæ are caught on their descending branches, fading with a rate of, as before said, about $0^{m}\cdot052$ per day. The story of galactic Novæ shows that the objects really have to be followed every night at maximum in order to get records of the highest brilliancy.

It is rather difficult to fix the correction to maximum light on account of incomplete observations of Andromeda Novæ. From the material at hand I have in different ways computed this correction at $0^{m}\cdot7$. Thus the real maximum brightness of the Novæ in question would be $16^{m}\cdot34$. Using the value of m_{π} (max.) previously derived for galactic Novæ, we obtain :

Distance of N.G.C. 224 = **1,400,000** light-years.
Dimensions „ „ „ = **60,000** „ „ .

The condensed parts of the nebula may exercise some absorption, which makes the Novæ apparently fainter and thus increases their distances. From the material at hand we see that there can scarcely be any appreciable amount of absorption (fig. 10).

From the distance derived, we find that the Nova S Andromedæ at the maximum reached the huge magnitude of -16. One may hesitate to accept such a luminosity. I think that we have an analogous case in the famous Nova B Cassiopeiæ of 1572. In conjunction with Mr. Humason at the Mount Wilson Observatory, I tried to identify the Nova. A discussion of Tycho Brahe's extensive measures of the position of the star showed the place to be accurate within $0'\cdot25$. The only star close to Tycho's position that may be the lost Nova is a star of photographic magnitude $13\cdot7$ and a spectrum of the Mb class. The spectrum shows giant characteristics, and as the difference between the present magnitude and the maximum magnitude is at least $18^{m}\cdot7$, we find that the maximum magnitude must have been about -16. The identification may be uncertain, but the spectral types rather suggest the star has been a Nova, as the present

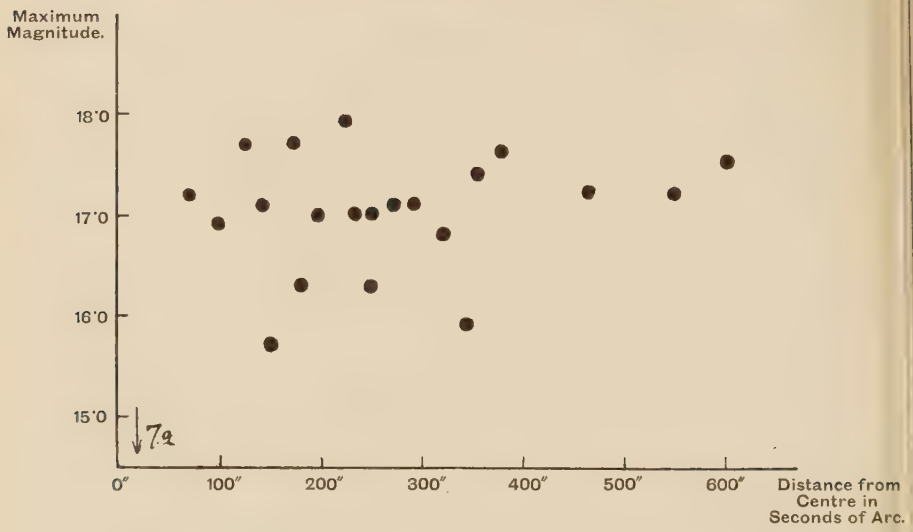

FIG 10.—Relation between magnitude at maximum and the distance of Novæ from centre in the Andromeda Nebula.

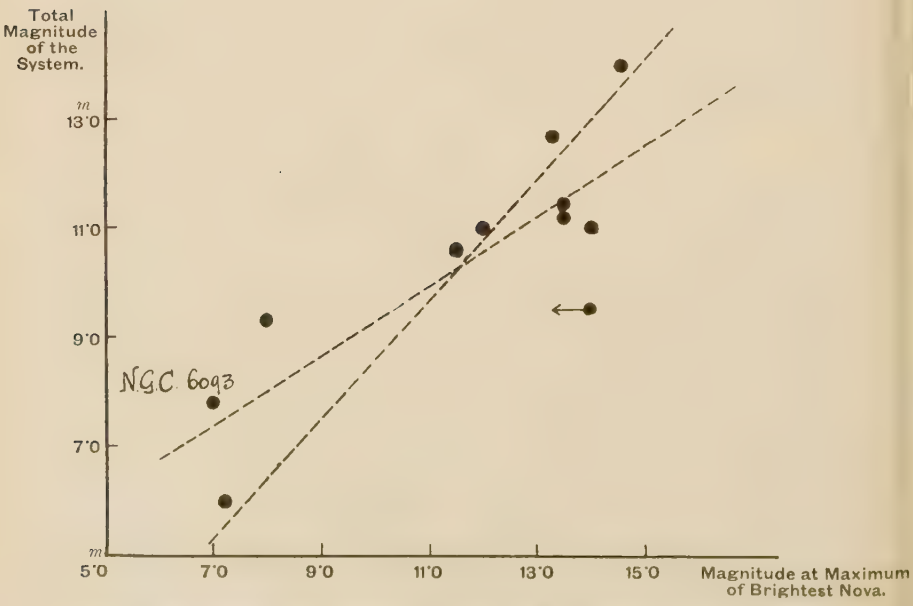

FIG. 11.—The Absolute Magnitude of "Upper Class" Novæ as compared with the Total Magnitude of the Stellar System in which they have appeared.

spectrum of Nova T Coronæ is Mb. It is true that in this case also bright lines are present, and, on the other hand, we have not found for certain any bright lines in the spectrograms of the star supposed to be Nova B Cassiopeiæ. The following table will show that neither Wolf Rayet spectrum nor bright lines are necessary features of former Novæ.

	Time since Maximum.	Spectral Class at Present.	Spectrum before Outburst.
Nova Cygni, 1920	5 years	P	
„ Aquilæ No. 3	7 „	P+Q	B–A
„ Geminorum No. 2	13 „	Q	
„ Persei No. 2	25 „	P	
„ T Aurigæ	33 „	Q	
„ Q Cygni	49 „	Continuous spectrum not planetary	
„ T Coronæ	59 „	Mb pec	
„ Ophiuchi, 1848	77 „	Early type; continuous spectrum not planetary.	
„ η Argus	91 „	F_5 pec	
„ P Cygni	331 „	B_4 pec	

An interesting analogy is the following. Supposing that B Cassiopeiæ actually reached the absolute magnitude -16, we see that this is practically the integrated absolute magnitude of the galactic system. Now, S Andromedæ was at maximum 7·2, which is of the same order of magnitude as the total or integrated magnitude, 4·9, of the nebula itself.

If we look up the cases where Novæ have appeared in spiral nebulæ we find that the same general rule will hold, as illustrated in fig. 11. The total magnitudes of the spirals in which Novæ are found have been estimated from the Franklin-Adams plates, which through the courtesy of the Astronomer Royal have been put at my disposal. The Cluster Nova T Scorpii has been included in the diagram as fitting the relation.

It is quite possible that we have to deal with two distinct classes of Novæ: one "upper class" having comparatively few members and reaching an absolute magnitude more or less equal to the absolute magnitude of the system in which they appear; one "lower class," in the mean 10 magnitudes fainter, but still reaching a very high luminosity at maximum. As the lower-class Novæ in most nebulæ other than the Andromeda will reach a maximum magnitude below 20^m, we have not been able to find the swarm of those Novæ, but have been restricted to the rather rare cases of "upper-class" objects. As pointed out by van Maanen, it is remarkable that no Nova has appeared in Messier 33,* and it is also singular that we have not seen any Nova in the Magellanic Clouds, although in this case incomplete records may explain a good deal.

But the Novæ are not restricted to spirals of the Andromeda type.

* The suspected object reported by Duncan proved to be no Nova. Hubble has signalled the discovery of a Nova in this spiral.

A Nova has appeared in the chaotic nebula N.G.C. 5253, and another one is recorded in the "globular nebula" N.G.C. 4486. The latter Nova reached a maximum magnitude of about 11·5.

The photographs of this nebula taken by Hubble show an extremely interesting phenomenon. At the border of the "nebulosity" there are numerous minute star-like points which evidently belong to the nebula, and which are not mere photographic effects. As the object is in a high galactic latitude these objects are presumably connected with the nebula. I suppose these stars to be the brightest ordinary stars that are to be found in a galactic system, and assign to them an absolute photographic magnitude of −7. As their apparent magnitude is 20, we see that the m_π (max.) of the Nova should be −15·5 in good agreement with the magnitude found for S Andromedæ. Using the diagram in fig. 11 and the above assumption, we find the following:

Distance	of N.G.C. 4486	=	**8,000,000**	light-years.
Dimensions	„ „ „	=	**6,000**	„ .

Thus the stars in this system are closely packed and the dimensions only a tenth of the dimensions of the Andromeda nebula. Anyhow, the object is too large to be comparable with a globular cluster, and it is more likely that we have in the globular nebulæ the forerunners of the Milky Way systems. It is very interesting to note that stars can be formed before the spiral arms appear.

Too much stress should not be laid on the fact that no spiral structure is seen in several "non-galactic" nebulæ. Sometimes it may only be a question of scale, in other cases, as in N.G.C. 4486, the physical constitution may be different. Still the so-called "globular nebulæ" do not seem to differ radically from the spiral nebulæ either with regard to spectral properties or radial motions. Between the typical spirals and the irregular non-galactic nebulæ, tentatively called the Magellanic Cloud type, there can be established a sequence of evolution. The spiral character probably is a feature characteristic of only a small number of the constituents of the spiral family.

(b) *Cepheids.*—Since the part of the present paper discussing the Novæ was written the discovery of Cepheids in the Andromeda nebula as well as in other spirals has been announced. Using Shapley's mean absolute magnitude of galactic Cepheids, Hubble derives 930,000 light-years as the distance of the Andromeda nebula. As Shapley's parallaxes are based on the proper motions of Boss, the question arises if the systematic correction of $+0''{\cdot}013 \cos\delta$ according to Kapteyn's suggestion has to be applied to the μ_δ's or not. In an extensive investigation Ralph E. Wilson has given the proper motions of 84 objects presumably belonging to the class of Cepheids, and discussed the mean parallax of these stars.*

For 51 objects in his list having a period larger than 1·9 days, there is no doubt as to the Cepheid nature. The proper motions have been used in the following way:—

* *A.J.*, **35**, 35, 1923.

"Globular" Nebula Messier 87 (N.G.C. 4486), photographed 1920 Feb. 26 with the 100-in. Reflector at Mt. Wilson.

From a convenient table computed by Strömberg the quantities :

$$a = 0{\cdot}2111[x_0 \sin\alpha - y_0 \cos\alpha];$$
$$b = 0{\cdot}2111[x_0 \cos\alpha \sin\delta + y_0 \sin\alpha \sin\delta - z_0 \cos\delta],$$

were taken and the equations of condition formed :

$$a\pi = \mu_\alpha \cos\delta$$
$$b\pi = \mu_\delta,$$

from which the mean parallax $\bar{\pi} = \dfrac{\Sigma(a\mu_\alpha \cos\delta + b\mu_\delta)}{\Sigma(a^2+b^2)}$ is obtained. If there is a correction μ_0 to be applied to the μ_δ's the parallax $\bar{\pi}_1$ becomes :

$$\bar{\pi}_1 = \bar{\pi} + \frac{\Sigma b\mu_0}{\Sigma(a^2+b^2)}.$$

The following mean parallaxes were obtained :

$0''{\cdot}00271$ (from μ_α's)
$0''{\cdot}00302$ (from μ_δ's).

The fact that the mean parallaxes are of the same order of magnitude rather suggests that the proper motions in declination are not affected by any large systematic error. The application of the correction $+0''{\cdot}013 \cos\delta$ will wipe out the value of the mean parallax from μ_δ, and that strengthens the conclusion reached by several astronomers that the whole effect should not be applied. The proper motions of the Cepheids allow very well the application of half the effect. Combining the results from both components of the motion, we get :

	$\bar{\pi}_1$.	
1.	$0''{\cdot}0029$	uncorrected
2.	$0''{\cdot}0020$	μ_δ's corrected with $+0''{\cdot}0065 \cos\delta$
3.	$0''{\cdot}0012$	,, ,, ,, $+0''{\cdot}0130 \cos\delta$.

The corresponding values of the distance of the Andromeda nebula are :

1. 620,000 light-years.
2. 880,000 ,,
3. 1,500,000 ,,

Some other uncertainties may be involved in the determination of the mean parallaxes of Cepheids. We have no definite idea as to the values of the radial velocities because, what is called rad. vel. of a Cepheid is what results from the assumption that the Cepheids are binary stars. Besides, one may have to consider the fact that the " velocity of the system " has proved to be a variable quantity (*e.g.* Y Sagittarii, Polaris, S Sagittæ).

The Cepheids in the Andromeda nebula are situated in the less dense parts of the nebula rather far from the centre. Thus these stars will probably not be affected by any appreciable absorption. The distance of the nebula will be obtained with considerable accuracy as soon as the absolute magnitudes of the galactic Cepheids have been derived with higher precision than is possible for the present.

(c) *Oepik's Method.*—An equation similar to the one given earlier in this paper connecting the central mass of a spiral with the linear rotational velocity of an object at a certain angular distance from the nucleus is easily derived.* The salient point in the method is to introduce the ratio I/M, representing the emission of radiation per unit mass, as determined from our galaxy. Oepik found a value for the distance of the Andromeda nebula of 1,500,000 light-years.

Using the measures of Pease for the rotation of the well-known object N.G.C. 4594, I have computed its distance according to Oepik's method and have obtained :

Distance of N.G.C. 4594 = **56,000,000** light-years.
Dimensions „ „ **110,000** „ .

The mass of the nebula is equal to $3.10^{10}\odot$.

(d) *Jeans' Method.*—This method is based upon a theoretical calculation according to Jeans' theory of spiral nebulæ of the mean distance between adjacent nuclei in the spiral arms. The comparison between the theoretical distances and the mean apparent separation of nuclei as estimated on photographs then gives the parallax. As pointed out before,† one can take the theory for granted, but question the estimated distances between the nuclei. The so-called nebulous background in the spiral arms is as a rule certainly not nebulosity, but the light from numerous stars apparently gathered together on account of the vast distance from which they are seen. There is no reason to believe that the 100-inch reflector, although a huge instrument, has the power to completely resolve the spiral arms into separate stars. Jeans' distances for spiral nebulæ show that these objects are not closer, but may very well be farther away. As I see this question, Jeans' theory for the formation of stars does not prevent the spirals being extended stellar systems, and the truth of the theory is not affected by the results here given concerning the distances of spirals.

Another difficulty in using Jeans' method is that the computed distances between the stars are confined to regions close to the nucleus where we cannot estimate the apparent separation at all. The condensations move farther apart from each other as they go outwards, and the mean distance varies considerably with the distance from the nucleus.

In a recent paper ‡ Jeans connects his method with the relation between luminosity and mass.

(e) *Counts of Stars and Estimates of Maximum Luminosity.*—In 1920 I used the swarms of " stars " shown on Mount Wilson photographs of Messier 33 for a determination of the distance. The simple assumption was that the resolved " stars " in the spirals had an absolute magnitude equal to the absolute maximum magnitude of stars in the galactic system. This quantity was supposed to be −7, which also is in harmony with recent determinations. The value thus assigned to Messier 33 was 1,500,000 light-years.§

* *Aph. J.*, **55**, 409, 1922.
† *M.N.*, **84**, 747, 1919.
‡ *M.N.*, **85**, 531, 1925.
§ *Publ. A.S.P.*, **33**, 324, 1921.

I. Total Spectrum of the Central Region of the Spiral Nebula M 33, taken with the Crossley Reflector and Keeler's quartz-spectrograph, 1921 Oct. 29. Exposure, 8 hours.

II. Total Spectrum of the Central Region of the Spiral Nebula M 33, taken with the Crossley Reflector and large quartz-spectrograph, 1921 Nov. 2, 4, and 5. Exposure, 30 hours.

That the star-like points in Messier 33 really are stars is proved by several slitless spectrograms taken by me using the Crossley reflector. The results of the study of these can be summed up in the statement that the so-called Nebula is composed of stars, clusters of stars, and also of nebulosity in more or less the same way as our Milky Way system.

The bright and dark nebulosities in Messier 33 and other spirals indicate other possibilities for determination of distances. In connection with a study of the corresponding formations in the Milky Way the distance of the spiral will also be discussed.

The star counts in Messier 33 permit us to fix the upper branch of the luminosity curve. Results which space does not permit us to give in detail of preliminary investigations of the resolved stars, the clusters, and the nebulous matter in Messier 33 yield distances for that nebula varying between 500,000 and 1,500,000 light-years.

Lastly, I discuss briefly the question as to the extension of the Universe.

In 1922 Dr. Charlier investigated whether an infinite world could exist according to the conceptions of Lambert.* He met the well-known objections of Olbers and Seeliger by showing that the total luminosity and the gravitational attraction become finite if a certain condition is fulfilled. A shortened proof will be given here.

Suppose that:

N_1 stars form together a galaxy G_1,
N_2 galaxies form together a galaxy G_2,

and so forth.

The individual systems are supposed to be spherical and uniformly distributed within the system of higher order; R_0 denotes the radius of a star and R_i the radius of a galaxy of the i^{th} order, which has a total mass M_i. M_0 is the mass of a star.

We then have:

$$M_i = N_i M_{i-1} = M_0 \prod_{i=1}^{\infty} N_i,$$

with the conditions:

$$\text{total attraction} = \sum_{i=1}^{\infty} \frac{M_i}{R_i^2} \neq \infty,$$

$$\text{total light} = \sum_{i=1}^{\infty} \frac{h_i}{R_i^2} \neq \infty.\dagger$$

The observer is supposed to be situated at the centre of each galaxy.

The first series converges if:

$$\frac{M_{i-1}}{R_i^2{}_{-1}} > \frac{M_i}{R_i^2}, \quad \text{or} \quad \frac{R_i^2}{R_i^2{}_{-1}} > \frac{M_i}{M_{i-1}}, \quad \text{or} \quad \frac{R_i}{R_{i-1}} > N_i^{\frac{1}{2}}.$$

The second series converges if:

$$\frac{h_i}{R_i^2} > \frac{h_{i-1}}{R_i^2{}_{-1}}, \quad \text{or} \quad \frac{h_i}{h_{i-1}} > \frac{R_i^2}{R_i^2{}_{-1}} > N_i.$$

* *Medd. fr. Lund's Astr. Observatorium*, No. 98, 1922.

† According to well-known theorems in Celestial Mechanics.

Thus if $\frac{R_i}{R_{i-1}} > N_i^{\frac{1}{2}}$ the sum of either series is finite.

As we certainly have absorption of light within the different galaxies a more general expression for the total light will be :

$$\sum_{i=1}^{\infty} \frac{h_i}{R_i^2} e^{-\lambda_i R_i},$$

and hence the convergence criterion :

$$\frac{R_i}{R_{i-1}} > N_i^{\frac{1}{2}} e^{\frac{\lambda}{2} R_i},$$

assuming $\lambda = \lambda_i = \lambda_{i-1}$, etc., and R_i large in comparison with R_{i-1}, etc.

This criterion further will be a little modified on account of an extinction effect. If we have an infinite number of galaxies in a certain direction, part of the light from the individuals will be extinguished.

Charlier identifies the spirals with the galaxies of the second order. He finds that the nearest spiral nebula must have an angular diameter amounting at the most to 5°·7. As the angular diameter of the Andromeda nebula is 2°, he concludes that the inequality may nearly be altered to the equality $\frac{R_2}{R_1} = N_2^{\frac{1}{2}}$.

Thus he finds :

Distance of N.G.C. **224** = **28** × *Diameter of galactic system.*

Supposing we know the place of our Sun with regard to the centre of the galactic system, we can easily, by assuming the galactic and the Andromeda Novæ to have equal absolute magnitudes, express the distance of the Andromeda nebula in terms of the diameter of the galactic system. Thus I have found :

Distance of N.G.C. **224** = **32** × *Diameter of galactic system.*

The good agreement between the two results obtained from suppositions so different is rather remarkable, and cannot very well be due to chance.

A comparison has also been made between the mean angular dimensions of the spirals and the mean angular distance between them. Mainly in accordance with a derivation by Pólya,* we compute the mean angular distance between n points scattered at random on a sphere having the surface 4π.

The probability that a point is situated within a certain area may be taken as proportional to the area. The probability that a point is farther away from one selected as origin than the spherical distance α is :

$$\frac{2\pi(1+\cos\alpha)}{4\pi} = \cos^2\frac{\alpha}{2}.$$

* *Astr. Nachr.*, **208**, 175, 19.

The probability that a point is farther away than $a+da$ is :

$$\frac{\cos^2(a+da)}{2}.$$

Hence the probability that it is farther away than a, but closer than $a+da$ is :

$$-d\cos^2\frac{a}{2}.$$

The probability that $n-1$ points are at a distance from the origin between a and $a+da$ is :

$$-d\cos^{2(n-1)}\frac{a}{2}.$$

The average value $\bar{a}$ is

$$=-\int_0^{\pi} ad\cos^{2(n-1)}\frac{a}{2}=\int_0^{\pi}\cos^{2(n-1)}\tfrac{1}{2}ada=2\int_0^{\frac{\pi}{2}}\cos^{2(n-1)}da=\tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n)}.$$

Applying Stirling's formula for $n!$ we obtain :

$$\bar{a}=\sqrt{\frac{\pi}{n-1}}.$$

Using the material in the N.G.C. outside the Milky Way, and assuming the following numerical values of the Herschelian symbols—

3″–4″,	10″–12″,	20″–30″,	50″–60″,	3′–4′,	8′–10′,	20′,
*e*S,	*v*S,	S–*c*S,	*p*S–*p*L,	*c*L–L,	*v*L,	*e*L,

the mean diameter $\bar{d}$ of 10,318 nebulæ is found to be 51″·8. As $\bar{a}$ is 3070″, we find :

$$\bar{a}=59\times\bar{d}.$$

A much better agreement with Charlier's theory is obtained by using the homogeneous material collected by Fath, from plates taken with the 60-inch reflector at Mount Wilson.* His material gives :

$$\bar{a}=\mathbf{28}\times\bar{\mathbf{d}}.$$

Thus we are for the second time led to practically the same numbers as Charlier. Our present knowledge as to the space-distribution of the stars and the spirals can be summed up in the statement :

Our Stellar system and the system of spiral nebulæ are constructed according to the conceptions expressed in the Lambert-Charlier cosmogony.

Whether galaxies of the third order exist we do not know. As it was said by Pease† that the appearance of N.G.C. 3379 suggested a cluster of nebulæ, I thought at first that globular nebulæ might be galaxies of a higher order. The results concerning the spectra and velocities of these objects show that they are not exceptions among the spirals, but probably forerunners of them. Besides, as the number of spirals

* *A.J.*, **28**, 86, 1914.
† *Mount Wilson Contr.*, **186**, 17, 1920.

is of the order of 10^6, the system of spiral nebulæ or the galaxies of the third order must be smaller than 8″. Thus they are not distinguishable from small spirals. For the present it is difficult to see how it will be possible to get a criterion for recognising such systems. The velocities of the G3's will be much larger than those of the G2's, but, of course, it is doubtful if spectroscopy ever will have the means of analysing the light emitted from the galaxies of higher order than the system of spiral nebulæ.

Upsala and Greenwich:
1925.

Naked-Eye Observations of Venus.

By J. K. Fotheringham, D.Litt.

The following observations are in continuation of those published by Professor Turner in *M.N.*, **84** (1924), 770–2. All altitudes are true, refraction being disregarded.

First Appearance in Evening.

1925 May 22. S. W. H. watched near Bicester (1° 10′ W., 51° 54′ N.). Numerous patches of cloud in N.W. rendered observation difficult. Venus not seen.
23–9. Too cloudy for observation.
30. S. W. H. saw Venus about $20^h\ 25^m$ G.M.T., and pointed it out to N. B. H.
31. S. W. H. saw Venus.

J. K. F. watched at Oxford from May 21 to May 30 and June 1 to June 5, but did not see Venus.

H. G. T. reported on May 31 that he had kept a look out for Venus at Dedham, but had had no luck as the sky was clouded.

The computed altitudes for S. W. H.'s observation on May 30 are:

	Time.	Alt. of Venus.	Alt. of Sun.
	h m	°	°
Venus seen . .	20 25	+4·0	−2·4
Venus set . .	20 57·7	..	−6·06

Stellar magnitude of Venus −3·4.

The observation is less striking than that of J. H. J. at the same phase on 1923, Nov. 7, but the Sun was nearer to the horizon at the setting of Venus than it was at the setting and rising respectively at last appearance in evening and first in morning recorded in 1924.

University Observatory, Oxford:
1925 *June* 6.

Note.—J. K. F. continued watching, but did not see Venus till June 16. He did not watch on June 15. S. W. H. saw Venus on June 15 well before sunset.

1925 *June* 29.

Nova Pictoris. By W. E. Cooke, Government Astronomer, Sydney.

We possess a catalogue plate covering the position of Nova Pictoris, taken on 1924 March 1. The exposures are 4 m., 2 m., and 13 s., and the plate is a good one. The check plate on a standard Harvard region, taken on the same night, shows all three images down to 11·15 magnitude, and two images (4 m. and 2 m.) down to 12·88 magnitude. We may assume that the catalogue plate has an equal scale.

A very careful examination of the réseau square which now contains the Nova shows the entire space (5′ square) to be devoid of any images, with the exception of a barely perceptible small smudge. On comparing the Nova with neighbouring stars on a recently-taken photograph, it is found that its position exactly corresponds with that of the little smudge, so in all probability it represents the Nova before the outbreak. I should estimate the magnitude on the Harvard scale to be about 14·2.

Observations with the transit circle on June 2, 3, 4 gave the mean position of the Nova for 1925·0 as

$$6^h\ 34^m\ 57^s\cdot01, \quad -62^\circ\ 34'\ 34''\cdot3.$$